THE GELFAND PROBLEM ON ANNULAR DOMAINS OF DOUBLE REVOLUTION WITH MONOTONICITY

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Abstract. We consider the following Gelfand problem

$$(P)_\lambda \qquad \begin{cases} -\Delta u = \lambda a(x) f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter and $f(u) = e^u$ or $f(u) = (u+1)^p$ where $p > 1$ and $a(x)$ is a nonnegative function with certain monotonicity (we allow $a(x) = 1$). Here Ω is an annular domain which is also a double domain of revolution. Our interest will be in the question of the regularity of the extremal solution. We obtain improved compactness because of the annular nature of the domain and we obtain further compactness under some monotonicity assumptions on the domain.

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1. Introduction

We are interested in the following Gelfand problem

$$(P)_\lambda \qquad \begin{cases} -\Delta u = \lambda a(x) f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(u) = e^u$ or $f(u) = (u+1)^p$ where $p > 1$ and where $\lambda > 0$ is a parameter and $a(x)$ is a nonnegative function with certain monotonicity. Here Ω is an annular domain which is also a domain of double revolution. Our interest will be in the question of the regularity of the extremal solution. There is a natural improvement of compactness on annular domains. As in [18] there is a further increase of compactness on annular domains with monotonicity. We now give a brief background on $(P)_\lambda$. We define a weak solution u of $(P)_\lambda$ to be function $u \in L^1(\Omega)$ so that $\lambda a f(u) \in L^1_\delta(\Omega) = L^1\big(\Omega, \delta(x)dx\big)$, $\delta(x) = dist(x, \partial\Omega)$ and

$$\int_\Omega (-\Delta\varphi) u \, dx = \int_\Omega \lambda a f(u) \varphi \, dx,$$

holds for any $\varphi \in C^2(\overline{\Omega})$, $\varphi = 0$ on $\partial\Omega$ (see [4]). It is well known ([4, 19, 21, 36]) that there exists a finite positive extremal parameter λ^* such that for any $0 < \lambda < \lambda^*$, problem $(P)_\lambda$ has a minimal classical solution u_λ, while no solution exists even in the weak sense for $\lambda > \lambda^*$. The branch $\{u_\lambda\}_{0<\lambda<\lambda^*}$ is increasing and the increasing pointwise limit $u^*(x) = \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ is a weak solution of $(P)_{\lambda^*}$, which is called the extremal solution, moreover the extremal solution is the unique weak solution of $(P)_{\lambda^*}$, see [29]. It is also well known that when f is convex then the minimal branch is stable in the sense that the operator $-\Delta - \lambda a f'(u_\lambda)$ has a positive first eigenvalue in $H_0^1(\Omega)$ and using the variational nature of this eigenvalue we arrive at

$$\int_\Omega \lambda a f'(u_\lambda)\phi^2 dx \leq \int_\Omega |\nabla \phi|^2 dx, \quad \forall \phi \in H_0^1(\Omega), \tag{1}$$

which typically is called the stability inequality. Given a solution u of $(P)_\lambda$ we denote the Morse index of u by the number of negative eigenvalues (counting multiplicity) of the linear operator $-\Delta - \lambda a f'(u)$ and hence a stable solution has Morse index zero.

1.1. **The case of $a = 1$.** The regularity properties of the extremal solution of problem $(P)_\lambda$ (in the case of $a = 1$) have been studied extensively in the literature, in particular after Brezis and Vázquez raised some open problems in [6, 5]. It was shown that it depends strongly on the dimension N, domain Ω and nonlinearity f. When $f(u) = e^u$ and Ω is an arbitrary smooth bounded domain, it is well known that $u^* \in L^\infty(\Omega)$ if $N \leq 9$ (see [19, 30]), while $u^*(x) = -2\log|x|$ and $\lambda^* = 2N - 4$ if $N \geq 10$ and Ω is the unit ball of $\mathbb{R}^N$ (see [6, 28]). The geometry of Ω can also play a role, for instance in [20] it is shown that if the domain is close enough to a ball and $N \geq 11$ then the extremal solution is unbounded. In the opposite direction they also prove in [20] that certain thin convex domains in large dimension can have a bounded extremal solution which was maybe unexpected. Our results in the current paper will be along the flavour of this second result.
It may be noted here that a comprehensive study of the above Gelfand-type problem in the asymptotically linear case, that is $\lim_{t\to\infty} \frac{f(t)}{t} := l \in (0,\infty)$ is given in [31], in particular, it is shown that when $\lambda = \lambda^*$ then there exists a classical solution of the problem if and only if $\lim_{t\to\infty} f(t) - lt < 0$. We also refer the interested reader to the survey article [2] for the existence of positive solutions of elliptic eigenvalue problems of the form $\mathcal{L}u = \lambda f(u)$ in Ω with $u = 0$ on $\partial\Omega$, where $\mathcal{L}$ is a strongly uniformly elliptic linear differential operator of second order with smooth real coefficients. Also, see [39] for bifurcation results on a more general class of problems $Lu + H(u) = \lambda u$, $u \in E$, where E is a Hilbert space, $L : E \to E$ is linear and $H \in C^1(E, E)$ is such that $H(0) = H'(0) = 0$.

In the case of $f(u) = (u+1)^p$ the Joseph-Lundgren exponent $p_{JL} = p_{JL}(N)$ plays a critical role where

$$p_{JL}(N) := \begin{cases} \infty & \text{if } N \leq 10, \\ \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-1)} & \text{if } N \geq 11. \end{cases}$$

In the case of $1 < p < p_{JL}$ the extremal solution is bounded and for $p > p_{JL}$ the extremal solution may be unbounded (for instance on the unit ball its unbounded), see [6]. Note $p_{JL} > p_s$ where $p_s := \frac{N+2}{N-2}$ (for $N \geq 3$) where $H_0^1(\Omega) \subset L^{p_s+1}(\Omega)$ is the critical Sobolev imbedding.

It is well known that regularity of solutions on bounded domains is closely related to Liouville theorems of related equations on the full space or half spaces via blow up arguments (see [15, 27, 26]). In the context of the polynomial problem we consider there exist positive stable solutions of

$$-\Delta v = v^p \quad \text{in } \mathbb{R}^N, \tag{2}$$

when $p > p_{JL}$. The related Liouville theorem would be: there are no positive stable solutions of (2) for $1 < p < p_{JL}$ which is exactly the same range of p where we have the extremal solution is bounded. These blow up arguments lend themselves more readily to questions relating the sequences of smooth solutions on bounded domains with constraints on the Morse index and apriori bounds, see [23, 24, 38] for related problems. The results in our current work easily extend to solutions with finite Morse index; see Remark 2 part 1.

The question regarding the regularity of the extremal solution for general f (under suitable minimal assumptions; smoothness, increasing, $f(0) = 1$, convex and superlinear at ∞) has been an extremely well studied problem see, [1, 7, 9, 10, 11, 12, 13, 14, 19, 21, 20, 23, 25, 30, 34, 35, 37, 40, 41]. A longstanding conjecture due to Brezis is whether the extremal solution is bounded provided $N \leq 9$. This conjecture was recently proved after 25 years in [8].

Remark 1. *We point out that our current work is not in the direction of extending known results for $a = 1$ to general a. Also we are not using conditions on a to increase compactness, as in the case of Hénon equation with $a(x) = |x|^\alpha$ in the unit ball B_1 centred at the origin. In our case we can add a function a provided it preserves the symmetry and, in the case of a monotonic domain, also preserves the monotonicity. The added compactness we are getting is coming from the annular nature of the domain and then further compactness is coming from the monotonicity of domain and function.*

1.2. **Domains of double revolution.** Unless explicity stated we are always assuming our domains will be domains of double revolution. We mention our motivation to study domains of double revolution originated from the work [12]. Consider writing $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ where $m, n \geq 1$ and $m + n = N$. We define the variables s and t by

$$s := \left\{x_1^2 + \cdots + x_m^2\right\}^{\frac{1}{2}}, \qquad t := \left\{x_{m+1}^2 + \cdots + x_N^2\right\}^{\frac{1}{2}}.$$

We say that $\Omega \subset \mathbb{R}^N$ is a *domain of double revolution* if it is invariant under rotations of the first m variables and also under rotations of the last n variables. Equivalently, Ω is of the form $\Omega = \{x \in \mathbb{R}^N : (s,t) \in U\}$ where U is a domain in $\mathbb{R}^2$ symmetric with respect to the two coordinate axes. In fact,

$$U = \left\{(s,t) \in \mathbb{R}^2 : x = (x_1 = s, x_2 = 0, ..., x_m = 0, x_{m+1} = t, ..., x_N = 0) \in \Omega\right\},$$

is the intersection of Ω with the (x_1, x_{m+1}) plane. Note that U is smooth if and only if Ω is smooth. We denote $\widehat{\Omega}$ to be the intersection of U with the first quadrant of $\mathbb{R}^2$, that is,

$$\widehat{\Omega} = \left\{(s,t) \in U : \ s > 0, \ t > 0\right\}. \tag{3}$$

Using polar coordinates we can write $s = r\cos(\theta)$, $t = r\sin(\theta)$ where $r = |x| = |(s,t)|$ and θ is the usual polar angle in the (s,t) plane.

The domains under the consideration will be annular domains with a certain monotonicity (or convexity) assumption in the polar angle θ. All domains will be bounded domains in $\mathbb{R}^N$ with smooth boundary unless otherwise stated. To describe the domains in terms of the above polar coordinates we will write

$$\widetilde{\Omega} := \left\{(\theta, r) : (s,t) \in \widehat{\Omega}\right\}. \tag{4}$$

Annular domains. We begin by considering an explicit annular domain in $\mathbb{R}^N$ and then we will generalize. The first example would be an annulus centred at the origin with inner radius R_1 and outer radius R_2; $\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$. Then we have $U = \{(s,t) : R_1^2 < s^2 + t^2 < R_2^2\}$ and finally we have $\widetilde{\Omega} = \{(\theta, r) : g_1(\theta) < r < g_2(\theta), \theta \in \left(0, \frac{\pi}{2}\right)\}$ where $g_1(\theta) = R_1$ and $g_2(\theta) = R_2$.

We can now consider a slightly more general version where the inner and outer boundaries are replaced with ellipsoids instead of balls. Take Ω to have outer boundary given by the ellipsoid

$$\sum_{k=1}^{m} \frac{x_k^2}{a^2} + \sum_{k=m+1}^{N} \frac{x_k^2}{b^2} = 1,$$

and the inner boundary given by

$$\sum_{k=1}^{m} \frac{x_k^2}{c^2} + \sum_{k=m+1}^{N} \frac{x_k^2}{d^2} = 1,$$

where $a, b, c, d > 0$ are chosen such that the resulting domain is an annular region. Note in this case we have

$$\widehat{\Omega} = \left\{(s,t) : s, t > 0,\ \frac{s^2}{a^2} + \frac{t^2}{b^2} < 1 \text{ and } \frac{s^2}{c^2} + \frac{t^2}{d^2} > 1\right\},$$

and

$$\widetilde{\Omega} = \left\{(\theta, r) : g_1(\theta) < r < g_2(\theta), \theta \in \left(0, \frac{\pi}{2}\right)\right\},$$

where the functions g_1 and g_2 are given by

$$g_1(\theta) = \frac{1}{\left(\frac{1}{c^2} + \sin^2(\theta)\left(\frac{1}{d^2} - \frac{1}{c^2}\right)\right)^{\frac{1}{2}}}, \quad g_2(\theta) = \frac{1}{\left(\frac{1}{a^2} + \sin^2(\theta)\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\right)^{\frac{1}{2}}}.$$

From this example we now introduce the idea of an *annular domain with monotonicity*. Consider the annular region in the (s,t) variables if we make the further restriction $c \le d < b \le a$; note we can consider this region as being obtained by starting with two concentric spheres in the (s,t) plane and vertically compressing the outer sphere and vertically stretching the inner one and then U is the region between the two deformed spheres. In terms of g_i note that g_1 is increasing on $(0, \frac{\pi}{2})$ and g_2 is decreasing on $(0, \frac{\pi}{2})$.

Definition 1. *We refer to a domain of double revolution in $\mathbb{R}^N$ with $N = m + n$ an annular domain if its associated domain $\widehat{\Omega}$ in the (s,t) plane in $\mathbb{R}^2$ is of the form*

$$\widetilde{\Omega} = \left\{(\theta, r) : g_1(\theta) < r < g_2(\theta), \theta \in \left(0, \frac{\pi}{2}\right)\right\} \tag{5}$$

in polar coordinates. Here $g_i > 0$ is smooth on $[0, \frac{\pi}{2}]$ with $g_i'(0) = g_i'(\frac{\pi}{2}) = 0$ and $g_2(\theta) > g_1(\theta)$ on $[0, \frac{\pi}{2}]$. We will call Ω an annular domain with monotonicity *if g_1 is increasing and g_2 is decreasing on $(0, \frac{\pi}{2})$.*

1.3. **Main results.** We begin by stating our assumptions on the $a(x)$ term.

$(\mathcal{A})$: Conditions on $a(x)$. We take the function a to be a continuous nonnegative function of (s,t) that is $a(x) = a(s,t)$. Moreover, we say that a satisfies $(\mathcal{A})$ if a is a continuously differentiable function with respect to (s,t) and $s a_t - t a_s \le 0$ in $\widehat{\Omega}$. We are allowing a to be zero in some regions of $\widehat{\Omega}$ but we are not using this to gain compactness; recall in the Hénon equation the term $|x|^\alpha$ allows one to gain some extra compactness on a ball centred at the origin.

Theorem 1. *The following assertions hold.*

(1) *(Annular domains without monotonicity) Suppose Ω is an annular domain in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ and $a = a(s,t)$ is nonnegative and Hölder continuous.*
   (a) *Suppose $f(u) = (u+1)^p$ and*
$$1 < p < \min\{p_{JL}(n+1), p_{JL}(m+1)\}, \tag{6}$$
   where $p_{JL}(k)$ is the Joseph-Lundgren exponent in dimension k. Then u^* is bounded.
   (b) *Suppose $f(u) = e^u$ and $m, n \leq 8$. Then u^* is bounded.*

(2) *(Annular domains with monotonicity) Suppose $\Omega \subset \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ is an annular domain with monotonicity in $\mathbb{R}^N$ with $n \leq m$ and a satisfies $(\mathcal{A})$.*
   (a) *Suppose $f(u) = (u+1)^p$ and*
$$1 < p < p_{JL}(n+1) = \max\{p_{JL}(n+1), p_{JL}(m+1)\}. \tag{7}$$
   Then u^* is bounded.
   (b) *Suppose $f(u) = e^u$, $\Omega \subset \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ is an annular domain with monotonicity in $\mathbb{R}^N$ and $n \leq 8$. Then u^* is bounded.*

2. Elliptic problems on domains of double revolution

We shall begin by providing some more background on quantities related to domains of double revolution that are essential in this work. Assume Ω is a domain of double revolution and v is a function defined on Ω that just depends on (s,t), then one has
$$\int_\Omega v(x)dx = c(m,n)\int_{\widehat{\Omega}} v(s,t)s^{m-1}t^{n-1}dsdt,$$
where $c(m,n)$ is a positive constant depending on n and m. Note that strictly speaking we are abusing notation here by using the same name; and we will continuously do this in this article. Given a function v defined on Ω we will write $v = v(s,t)$ to indicate that the function has this symmetry. Define
$$H^1_{0,G} := \left\{u \in H^1_0(\Omega) : gu = u \quad \forall g \in G\right\},$$
where $G := O(m) \times O(n)$ where $O(k)$ is the orthogonal group in $\mathbb{R}^k$ and $gu(x) := u(g^{-1}x)$.

To solve equations on domains of double revolution one needs to relate the equation to a new one on $\widehat{\Omega}$ defined in (3). Suppose Ω is a domain of double revolution and f is a function defined on Ω with the same symmetry (ie. $gf(x) = f(g^{-1}x)$ all $g \in G$). Suppose that $u(x)$ solves
$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$
Then $u = u(s,t)$ and u solves
$$-u_{ss} - u_{tt} - \frac{(m-1)u_s}{s} - \frac{(n-1)u_t}{t} = f(s,t) \text{ in } \widehat{\Omega}, \tag{9}$$

with $u = 0$ on $(s,t) \in \partial\widehat{\Omega} \backslash (\{s = 0\} \cup \{t = 0\})$. If u is sufficiently smooth then $u_s = 0$ on $\partial\widehat{\Omega} \cap \{s = 0\}$ and $u_t = 0$ on $\partial\widehat{\Omega} \cap \{t = 0\}$ after considering the symmetry properties of u.

2.1. **Improved compactness via annular domains and via monotonicity.** We now define a convex set K that we work on to give increased compactness. We mention that this idea of restricting functions to ones which are monotonic in an angle, to improve compactness, is coming from the work of [3] where they examined some Lane-Emden type equations in an annulus. The current setting of nonradial annular domains with (and without) monotonicity is coming from [18] where this monotonicity is combined with an abstract nonsmooth variational principle, due to A. Moameni, to obtain nontrivial solutions of supercritical problems, see [16, 17, 32, 33]. Here we are not considering this variational principle since we are only considering the regularity of the minimal solutions which are known to exist; we could also consider questions of regularity of solutions with finite Morse index again without considering the variational principle.

We define K by

$$K = K(m,n) := \left\{ 0 \le u \in H^1_{0,G}(\Omega) : su_t - tu_s \le 0 \text{ a.e. in } \widehat{\Omega} \right\}, \tag{10}$$

and note we can rewrite K as functions u such that if we write (s,t) in terms of polar coordinates we have $u_\theta \le 0$ in $\widetilde{\Omega}$ defined in (4).

Theorem A. *([18]) (Imbeddings for annular domains) Let Ω denote an annular region in $\mathbb{R}^N$.*

(1) *(Imbedding without monotonicity) Suppose Ω has no monotonicity and*
$$1 \le p < \min\left\{ \frac{2(n+1)}{n-1}, \frac{2(m+1)}{m-1} \right\}.$$
Then $H^1_{0,G}(\Omega) \subset\subset L^p(\Omega)$ with the obvious interpretation in the case of $m = n = 1$.

(2) *(Imbedding with monotonicity) Suppose Ω is an annular domain with monotonicity, $n \le m$ and*
$$1 \le p < \frac{2(n+1)}{n-1} = \max\left\{ \frac{2(n+1)}{n-1}, \frac{2(m+1)}{m-1} \right\}.$$
Then $K \subset\subset L^p(\Omega)$ with the obvious interpretation if $n = 1$.

The idea of the imbedding is that on a annular domain one expects and improved imbedding and this is what is given in the first part of Theorem A. If we have monotonicity (in the right direction) we can further improve the imbedding and that is given in the second part of the theorem. The above imbedding together with a new nonsmooth variational principle enabled the authors in [18] to deal with supercritical problems on annular doamins variationally, and to prove the multiplicity of positive solutions for such problems.

3. Proofs

Lemma 1. *Let Ω denote an annular domain with monotonicity and a satisfies $(\mathcal{A})$. Then for all $0 < \lambda < \lambda^*$ one has $u_\lambda \in K$.*

Proof. There are multiple ways to try and prove the result. We will use directly the linear iteration that is used to construct the minimal solution to prove the result. An alternate way would be to use the fact that the linearized operator satisfies the maximum principle. For $k \ge 0$ we set

$$\begin{cases} -\Delta u^{k+1} = \lambda a(x) f(u^k) & \text{in } \Omega, \\ u^{k+1} = 0 & \text{on } \partial\Omega, \end{cases}$$

with $u_0 = 0$. Then for $0 < \lambda < \lambda^*$ one has that $u^k(x) \nearrow u_\lambda(x)$ as $k \to \infty$. Writing this in terms of the (s,t) coordinates we have

$$\left\{ \begin{array}{ll} -u_{ss}^{k+1} - u_{tt}^{k+1} - \frac{(m-1)}{s} u_s^{k+1} - \frac{(n-1)}{t} u_t^{k+1} = \lambda a(s,t) f(u^k) & \text{in } \widehat{\Omega}, \end{array} \right.$$

with $u^{k+1} = 0$ on the curved portions of $\partial\widehat{\Omega}$ and $\partial_\nu u = 0$ on the remaining portions (ie. on $\Gamma = \partial\widehat{\Omega} \cap (\{s = 0\} \cup \{t = 0\})$). We now discuss the Neumann boundary condition since this relies on the symmetry and smoothness of the function.

Since $u^{k+1} = u^{k+1}(s,t)$ is the restriction to the first quadrant of (x_1, x_{m+1}) plane of an even $C^{1,\alpha}$ function in x_1 and x_{m+1} we see that $u_s^{k+1}, u_t^{k+1} \in C^{0,\alpha}(\overline{\Omega})$. This is sufficient regularity for u_s^{k+1} and u_t^{k+1} to give the desired boundary conditions on $\{s = 0\}$ and $\{t = 0\}$ portions of $\partial\widehat{\Omega}$ respectively. Define $v^k = s u_t^k - t u_s^k = u_\theta^k$ and then note that

$$L(v^{k+1}) = \lambda f'(u^k) v^k + \lambda f(u^k)(s a_t - t a_s) \quad \text{in } \widehat{\Omega},$$

where

$$L(\phi) = -\phi_{ss} - \phi_{tt} - \frac{(m-1)\phi_s}{s} - \frac{(n-1)\phi_t}{t} + \frac{(m-1)\phi}{s^2} + \frac{(n-1)\phi}{t^2},$$

and note that, up to issues with singularities, L satisfies a maximum principle on $\widehat{\Omega}$; for the time being we assume there are no issues here.

So the idea is to show that $v^k \le 0$ in $\widehat{\Omega}$ for all $k \ge 0$. Suppose $v^k \le 0$ in $\widehat{\Omega}$ for some $k \ge 0$. We now claim that $v^{k+1} = u_\theta^{k+1} \le 0$ on $\partial\widehat{\Omega}$. To see that $v^{k+1} = 0$ on the portions of the boundary that correspond to $\{s = 0\}$ and $\{t = 0\}$ we use the boundary conditions for u_s^{k+1} and u_t^{k+1}. For the curved portions of the boundary note that we have $u^{k+1} \ge 0$ in Ω and $u^{k+1} = 0$ on $\partial\Omega$ and hence $\partial_\nu u^{k+1} \le 0$ on $\partial\Omega$ and one can relate this the derivative in θ; instead of doing this it is much easier to view the pde in $\widetilde{\Omega}$ and then note that since $u^{k+1} \ge 0$ we immediately get $u_\theta^{k+1} \le 0$ on the curved portions after noting the monotonicity assumptions on g_i. Since we were assuming that we could apply the maximum principle we would then get $v^{k+1} \le 0$ in $\widehat{\Omega}$. To complete the proof we just need to show that we can start the iteration process; since $u^0 = 0$ we have $-\Delta u^1 = \lambda a$ in Ω and hence $L(v^1) = \lambda(s a_t - t a_s) \le 0$ in $\widehat{\Omega}$ and hence we can start the iteration after considering the boundary conditions for u^1.

We now give more details about the maximum principle argument since there are issues with the operator having singularities. Let $\varepsilon > 0$ be small and consider $\psi = (v^{k+1} - \varepsilon)_+$ and note $\psi = 0$ near $s = 0$ and $t = 0$. Now set $d\mu(s,t) = s^{m-1} t^{n-1} ds dt$ and note

$$\begin{aligned} 0 &\geq \int_{\widehat{\Omega}} \left\{ \lambda f'(u^k) v^k + \lambda (s a_t - t a_s) f(u^k) \right\} \psi d\mu \\ &= \int_{\widehat{\Omega}} L(v^{k+1}) \psi d\mu \\ &= \int_{\widehat{\Omega}} (\psi_s^2 + \psi_t^2) d\mu + \int_{\widehat{\Omega}} \left(\frac{(m-1) v^{k+1} \psi}{s^2} + \frac{(n-1) v^{k+1} \psi}{t^2} \right) d\mu \\ &\geq \int_{\widehat{\Omega}} \left(\psi_s^2 + \psi_t^2 + \psi^2 \left\{ \frac{(m-1)}{s^2} + \frac{(n-1)}{t^2} \right\} \right) d\mu \end{aligned}$$

and hence $\psi = 0$ a.e. in $\widehat{\Omega}$ and hence $v^{k+1} \leq \varepsilon$ a.e. in $\widehat{\Omega}$ which gives us the desired result after noting $\varepsilon > 0$ is arbitrary.

Following the computation in [12] we have

$$u_s^{k+1} = \sum_{i=1}^{m} u_{x_i}^{k+1} \frac{x_i}{s} \quad \text{and} \quad u_t^{k+1} = \sum_{i=m+1}^{N} u_{x_i}^{k+1} \frac{x_i}{t}, \tag{11}$$

and since $u_{x_i}^{k+1} \to (u_\lambda)_{x_i}$ in $C^{0,\delta}(\overline{\Omega})$ we have $u_s^{k+1} \to (u_\lambda)_s$ and $u_t^{k+1} \to (u_\lambda)_t$ a.e. in $\widehat{\Omega}$. From this we can conclude that $s(u_\lambda)_t - t(u_\lambda)_s = (u_\lambda)_\theta \leq 0$ a.e. in $\widehat{\Omega}$ which gives us the desired monotonicity.

□

Remark 2. (1) *For the proof of Theorem 1 (2a) we will use a blow up argument since for the proof of Theorem 1 (2b) we used an iteration argument and the improved imbeddings. Either proof works but we decided to use both methods for the interest of the reader. Note if one wanted to extend these results to solutions with finite Morse index the natural approach would be the blow up approach since the only difference would be the need of Liouville theorems for finite Morse index solutions (which are available under the same restrictions on the parameters).*

(2) *Recall the concentric ellipsoids can be thought of as a prototypical annular domain with monotonicity under the correct assumptions on the parameters. Note that the moving plane method would show that there is a strip near the outer boundary where the solution cannot attain its maximum. This relies on the convexity of the domain that one would obtain if they considered the domain induced by the outer boundary. One should note that the monotonicity assumptions on g_2 does not induce this convexity. For example take $g_2(\theta) = 2 + \cos(2\theta)$ and note this satisfies the assumptions on g_2 and yet it does not give the desired convexity condition.*

In our proof using blow up analysis we will assume that we have this added convexity to rule out a case where the maximums are close to the outer boundary. This case can easily be handled with the blow up analysis but since it is fairly standard we chose to omit it and more concentrate on the other cases.

Proof of Theorem 1. (2b). Here we assume $a(x) = 1$, but the exact same proof works if we assume $a(x)$ satisfies the assumptions stated earlier. We will use an approach that avoids having to use L^p regularity theory in $\widehat{\Omega}$. For $\frac{\lambda^*}{2} < \lambda < \lambda^*$ set $u = u_\lambda$ the minimal solution. All estimates will be uniform in λ and hence we can pass to the limit in λ. We assume $2 \leq n \leq 8$ (for $n = 1$ we use a different argument) and $n \leq m$. Then standard test function argument shows that there is some C (indepedent of λ) such that

$$\int_\Omega e^{(2\tau+1)u_\lambda} dx \leq C,$$

for all $1 < \tau < 2$ and hence we have a uniform (in λ) $L^T(\Omega)$ estimate on e^{u_λ} for all $T < 5$. For the readers convenience we now give the argument. Put $\phi = e^{\tau u_\lambda} - 1$ into the stability inequality (1) to arrive at

$$\int_\Omega \lambda e^{u_\lambda}(e^{\tau u_\lambda} - 1)^2 dx \leq \tau^2 \int_\Omega e^{2\tau u_\lambda} |\nabla u_\lambda|^2 dx,$$

where $1 < \tau < 2$. Now mulitply $(P)_\lambda$ by $e^{2\tau u_\lambda} - 1$ and integrate by parts to arrive at

$$\tau^2 \int_\Omega e^{2\tau u_\lambda} |\nabla u_\lambda|^2 dx = \frac{\tau\lambda}{2} \int_\Omega e^{u_\lambda} \left(e^{2\tau u_\lambda} - 1\right) dx,$$

and now you can equate the terms with the gradient to arrive at

$$\int_\Omega e^{u_\lambda} \left(e^{\tau u_\lambda} - 1\right)^2 dx \leq \frac{\tau}{2} \int_\Omega e^{u_\lambda} \left(e^{2\tau u_\lambda} - 1\right) dx,$$

and now one can expand the integrals and collect like terms. Note the highest order term will be $e^{(2\tau+1)u_\lambda}$ and we can use Hölder's inequality to control the lower order terms provided $\frac{\tau}{2} < 1$ or $\tau < 2$.

Fix $\gamma_0 > \frac{1}{2}$ and $\tau > 0$ such that

$$\tau\gamma_0 > \frac{n-1}{4} \quad \text{and} \quad \tau\gamma_0 < 2.$$

For $k \geq 0$ we set

$$\gamma_{k+1} := \frac{(n+1)\gamma_k}{n-1} - \frac{1}{2\tau}.$$

Under the assumptions on γ_0 and τ we see that $\gamma_k \nearrow \infty$ as $k \to \infty$. Note we have $\gamma_k > \gamma_0 > \frac{1}{2}$ for all $k \geq 1$. Define $v = v_\tau = e^{\tau u} - 1$.

Claim. Given $k \geq 0$ and suppose the right hand side of (12) is finite. Then (12) holds where

$$\left(\int_\Omega v^{2\gamma_{k+1}+\frac{1}{\tau}} dx\right)^{\frac{n-1}{n+1}} \leq \frac{C_n \lambda \tau \gamma_k^2}{2\gamma_k - 1} \int_\Omega (v+1)^{1+\frac{1}{\tau}} v^{2\gamma_k - 1} dx. \tag{12}$$

So note if there is some $C > 0$ such that

$$\int_\Omega v^{2\gamma_0 + \frac{1}{\tau}} dx \leq C,$$

uniformly in λ then for all $k \geq 0$ there is some D_k such that

$$\int_\Omega v^{2\gamma_{k+1}+\frac{1}{\tau}} dx \leq D_k, \tag{13}$$

uniformly in λ.

We now prove the claim. First note that v satisfies

$$-\Delta v = -e^{\tau u}\tau^2 |\nabla u|^2 + \lambda\tau e^{(\tau+1)u} \leq \lambda\tau(v+1)^{1+\frac{1}{\tau}} \quad \Omega,$$

with $v=0$ on $\partial\Omega$; in the last line we dropped the quadratic in $|\nabla u|$ term and rewrote the remaining term in terms of v. For $\varepsilon>0$ set $\phi(x)=v(x)^{2\gamma_k-1}-\varepsilon^{2\gamma_k-1}$ for $v>\varepsilon$ and 0 otherwise and note this is a nonnegative $H_0^1(\Omega)$ test function and hence testing the equation for v (and sending $\varepsilon \searrow 0$) gives

$$(2\gamma_k-1)\int_\Omega v^{2\gamma_k-2}|\nabla v|^2dx \le \tau\lambda\int_\Omega (v+1)^{1+\frac{1}{\tau}}v^{2\gamma_k-1}dx,$$

which we can rewrite as

$$\int_\Omega |\nabla v^{\gamma_k}|^2dx \le \frac{\tau\lambda\gamma_k^2}{2\gamma_k-1}\int_\Omega (v+1)^{1+\frac{1}{\tau}}v^{2\gamma_k-1}dx.$$

Now since $v\in K$ we can show that $v^{\gamma_k}\in K$ and hence we can use Theorem A to see that

$$\left(\int_\Omega v^{\frac{\gamma_k 2(n+1)}{n-1}}dx\right)^{\frac{n-1}{n+1}} \le \frac{C_n\tau\lambda\gamma_k^2}{2\gamma_k-1}\int_\Omega (v+1)^{1+\frac{1}{\tau}}v^{2\gamma_k-1}dx,$$

and hence we have

$$\left(\int_\Omega v^{2\gamma_{k+1}+\frac{1}{\tau}}dx\right)^{\frac{n-1}{n+1}} \le \frac{C_n\tau\lambda\gamma_k^2}{2\gamma_k-1}\int_\Omega (v+1)^{1+\frac{1}{\tau}}v^{2\gamma_k-1}dx,$$

since $\frac{\gamma_k 2(n+1)}{n-1}=2\gamma_{k+1}+\frac{1}{\tau}$ and this completes the proof of the claim. We now show we can start the process at $k=0$ and so its sufficient to show that

$$\int_\Omega (v+1)^{2\gamma_0+\frac{1}{\tau}}dx\le C,$$

uniformly in λ. Note that

$$(v+1)^{2\gamma_0+\frac{1}{\tau}}=e^{(2\tau\gamma_0+1)u},$$

and note under the assumptions on γ_0 and τ we have $2\tau\gamma_0+1<5$ and hence we have the desired integral is uniformly bounded in λ. From this we see that for all $1<T<\infty$ we have v is bounded in $L^T(\Omega)$ uniformly in λ. Set $w=w_\lambda$ denote a solution of $-\Delta w=\lambda\tau(v+1)^{1+\frac{1}{\tau}}$ in Ω with $w=0$ on $\partial\Omega$ and note $0\le v\le w$. From the $L^T(\Omega)$ bound on v we see that w is bounded in $L^\infty(\Omega)$ uniformly in λ and hence we have the same for v and from this we get the same for u.

(2a.) Again we assume $a(x)=1$ for the proof, but the same proof works for more general $a(x)$. Suppose the result is false and hence there is some $\lambda_k\nearrow\lambda^*$ such that $u^k=u_{\lambda_k}$ is such that $\|u^k\|_{L^\infty}=T_k\to\infty$. There is some $(s_k,t_k)\in\overline{\widehat{\Omega}}$ such that $u^k(s_k,t_k)=T_k$ and note by monotonicity assumption on u^k we have $t_k=0$. We will now assume the outer boundary has some added convexity to rule out the maximums being attained near the boundary; under the assumptions we have imposed this is not true but the proof could easily be adjusted to handle this case; see Remark 2 part 2. So with this assumption we have $(s_k,0)$ bounded away from the outer boundary; so there is some $\delta_0>0$ such that $s_k<g_2(0)-\delta_0$ for all k. We now consider the three cases:

(i) $(s_k-g_1(0))T_k^{\frac{p-1}{2}}\to\infty$,
(ii) $(s_k-g_1(0))T_k^{\frac{p-1}{2}}\to\gamma\in(0,\infty)$,
(iii) $(s_k-g_1(0))T_k^{\frac{p-1}{2}}\to 0$.

For $r_k > 0$, to be determined later, set

$$v^k(s,t) = \frac{u^k(s_k + r_k s, r_k t)}{T_k}, \quad (s,t) \in \widehat{\Omega}_k = \{(s,t) : (s_k + r_k s, r_k t) \in \widehat{\Omega}\}.$$

Then a computation shows that $v^k = v^k(s,t)$ satisfies

$$-v^k_{ss} - v^k_{tt} - \frac{(m-1)r_k v^k_s}{s_k + r_k s} - \frac{(n-1)v^k_t}{t} = \lambda_k r_k^2 T_k^{p-1}(v^k + T_k^{-1})^p \quad \text{in } \widehat{\Omega}_k, \quad (14)$$

with $v^k_t = 0$ on $\partial\widehat{\Omega}_k \cap \{t=0\}$ and $v^k_s = 0$ on $\partial\widehat{\Omega}_k \cap \{s = \frac{-s_k}{r_k}\}$ and $v^k = 0$ on the remainder of the boundary.

Case (i). Take r_k such that $r_k^2 T_k^{p-1} = 1$ and so $r_k \to 0$. We suppose we are in the case of s_k bounded away from $g_1(0)$. In this case its clear that for any $R > 0$ (large) there is some k_R such that for all $k \geq k_R$ we have $(-R,R) \times (0,R) \subset \widehat{\Omega}_k$. Using a standard compactness argument we can pass to the limit in (14) to find some $0 \leq v \leq 1$ a solution of

$$-v_{ss} - v_{tt} - \frac{(n-1)v_t}{t} = \lambda^* v^p \quad \text{in } \mathbb{R} \times (0,\infty), \quad (15)$$

with $v(0) = 1$ and $v_t = 0$ on $t = 0$. Note instead of using boundary regularity one could extend evenly in t and then use interior regularity to get the needed estimates to pass to the limit near $t = 0$. Note we can view this equation as being satisfied in $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ after noting we can view the function v as radial in the t coordinate in $\mathbb{R}^n$. Since u^k is a stable solution of $(P)_{\lambda_k}$ in Ω one has

$$\int_{\widehat{\Omega}} (\phi_s^2 + \phi_t^2) d\mu \geq \int_{\widehat{\Omega}} \lambda_k p (u^k)^{p-1} \psi^2 d\mu \quad \forall \phi \in H^1_{0,G}(\Omega),$$

where, as before, $d\mu = s^{m-1} t^{n-1} ds dt$. By using a change of variables in this stability inequality one can show that v^k is a stable solution of (14) and then one can pass to the limit in this inequality to see that v is a nonzero semi stable solution of (15). But we now recall that $1 < p < p_{JL}(n+1)$ which gives us the needed contradiction.

We now suppose we are still in case (i) but $s_k \searrow g_1(0)$ (by the boundary condition on the inner boundary we know that $s_k > g_1(0)$ and we already know s_k is bounded away from $g_2(0)$. We now suppose there is the maximal $\tau_k > 0$ such that

$$Q_k = \left(\frac{s_k + g_0(0)}{2}, g_2(0) - \delta_0 \right) \times (0, \tau_k) \subset \widehat{\Omega}.$$

If τ_k is bounded away from zero the problem becomes easier so lets assume $\tau_k \to 0$. Then for large enough k we can assume the upper left corner of this rectangle hits the inner boundary of $\widehat{\Omega}$. Near $(s,t) = (g_1(0), 0)$ we can write $s = h(t)$ for some h smooth and $h(0) = g_1(0)$ and $h'(0) = 0$. By the mean value theorem there is some $\widehat{\tau}_k \in (0, \tau_k)$ such that

$$h'(\widehat{\tau}_k)\tau_k = \frac{g_1(0) + s_k}{2} - g_1(0) = \frac{s_k - g_1(0)}{2},$$

and hence

$$|h'(\widehat{\tau}_k)| \left| \frac{\tau_k}{r_k} \right| = \frac{|s_k - g_1(0)| T_k^{\frac{p-1}{2}}}{2} \to \infty,$$

and since $|h'(\widehat{\tau}_k)| \to 0$ we see that $\frac{\tau_k}{r_k} \to \infty$. We now define

$$\widehat{Q}_k := \{(s,t) : (s_k + r_k s, r_k t) \in Q_k\} \subset \widehat{\Omega}_k,$$

and note we can write this as

$$\widehat{Q}_k = \left(\frac{g_1(0) - s_k}{2r_k}, \frac{g_2(0) - s_k - \delta_0}{r_k} \right) \times \left(0, \frac{\tau_k}{r_k} \right),$$

and note that $\widehat{Q}_k \to \mathbb{R} \times (0,\infty)$. The rest of argument follows as in the case of s_k bounded away from $g_1(0)$.

Case (ii). In this case we have $s_k \searrow g_1(0)$. In this case we take $r_k = s_k - g_1(0)$ and then note that $\widehat{\Omega}_k \to \{(s,t) : s > -1, t > 0\}$. Using boundary regularity argument on (14) we can pass to the limit to find a classical solution of

$$-v_{ss} - v_{tt} - \frac{(n-1)v_t}{t} = \lambda^* \gamma^2 v^p \quad \text{in } (s,t) \in (-1,\infty) \times (0,\infty), \tag{16}$$

with $v = 0$ on the left boundary and with $v_t = 0$ on the bottom boundary. Also recall we have $0 \le v \le 1$ with $v(0) = 1$. By extending evenly in t we have a positive solution on a half space. We can now apply Liouville results of either [24] or [22] to obtain the needed contradiction.

Case (iii). In this case we again we have $s_k \searrow g_1(0)$ and we take $r_k = s_k - g_1(0)$. We can argue as in case (ii) (the only difference is $\gamma = 0$) to arrive at a classical solution of

$$-v_{ss} - v_{tt} - \frac{(n-1)v_t}{t} = 0 \quad \text{in } (s,t) \in (-1,\infty) \times (0,\infty), \tag{17}$$

with $v = 0$ on the left boundary and with $v_t = 0$ on the bottom boundary. Also recall we have $0 \le v \le 1$ with $v(0) = 1$. We can extend in t evenly to arrive at a nonconstant solution v of $\Delta v(x) = 0$ which attains its maximum at an interior point which contradicts the strong maximum principle.

(1a) and (1b). In this case we are not assuming any monotonicity of the domain. In the case of the polynomial nonlinearity we can perform a similar blow up argument. Here the solutions are not monotonic in θ and so there are extra cases to consider for the limiting problem. This will introduce the minimum of the Joseph-Lundgren exponents in the appropriate dimensions. For the exponential nonlinearity one can use a proof similar to the monotonic one we used but they would need to use the imbedding without monotonicity in place of the imbedding we used. Alternatively one could use a blow up argument for the exponential. □

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