Singular solutions of a p-Laplace equation involving the gradient

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Abstract

In this article we obtain positive singular solutions of

$$\begin{cases} -\Delta_p u = |\nabla u|^q & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{1}$$

where Ω is a small C^2 perturbation of the unit ball in $\mathbb{R}^N$. For $\frac{(p-1)N}{N-1} < q < p < N$ we prove that if Ω is a sufficiently small C^2 perturbation of the unit ball there exists a singular positive weak solution u of (1). For other ranges of p and q we prove the existence of Hölder continuous positive solution (with optimal regularity) on a C^2 perturbation of the unit ball.

1 Introduction

In this work we are interested in obtaining positive singular solutions of

$$\begin{cases} -\Delta_p u(y) = & C|\nabla u(y)|^q & y \in \Omega, \\ u = & 0 & y \in \partial\Omega, \end{cases} \tag{2}$$

where Ω is a small C^2 perturbation of the unit ball in $\mathbb{R}^N$ and where $C > 0$ is a constant. Note we can rewrite this as

$$0 = |\nabla u|^2 \Delta u + \frac{(p-2)}{2} \nabla u \cdot \nabla |\nabla u|^2 + C|\nabla u|^{q-p+4} \qquad y \in \Omega, \tag{3}$$

with $u = 0$ on $y \in \partial\Omega$. We can write this in terms of the components as

$$0 = \left(|\nabla u|^2 \Delta u + (p-2) \sum_{i,j=1}^{N} u_{y_i} u_{y_j} u_{y_i y_j} \right) + C|\nabla u|^{q-p+4}, \quad y \in \Omega. \tag{4}$$

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Note that we can re-write the equation as $-\Delta_p u - a(x)\cdot \nabla u = 0$ in Ω with $u = 0$ on $\partial\Omega$ where $a(x) = |\nabla u|^{q-2}\nabla u$ and hence if u sufficiently smooth we see that $a(x)$ should be sufficiently smooth so as to apply the maximum principle; hence the only solution should be $u = 0$. From this informal argument we expect the only way to obtain a positive solution is for the solution to be somewhat singular. The following example gives an explicit solution on the puncture of the unit ball. Our approach will be to perturb an explicit solution on the ball.

Example 1. *Let B_1 denote the unit ball centered at the origin in $\mathbb{R}^N$.*

1. *Let $1 < p < N$, $\frac{(p-1)N}{N-1} < q < p$ and define $w(r) := r^{-\sigma} - 1$ where $\sigma := \frac{p-q}{q-p+1}$ and*

$$C := \frac{(N-1)(q-p+1) - (p-1)}{(q-p+1)\sigma^{q-p+1}}. \tag{5}$$

Then u is a singular weak solution of (2) with $\Omega = B_1$. Note the restriction $p-1 < q < p$ forces $\sigma > 0$ and the further restriction forces $C > 0$.

2. *Let $q > max\left\{p, \frac{N(p-1)}{N-1}\right\}$ and define $u(r) := 1 - r^{\sigma}$ where $\sigma := \frac{q-p}{q-p+1}$ and*

$$C := \frac{(N-1)(q-p+1) - (p-1)}{(q-p+1)\sigma^{q-p+1}}. \tag{6}$$

Then u is a positive Hölder continuous weak solution of (2) with $\Omega = B_1$. Note the restriction $p < q$ forces $\sigma > 0$ and the further restriction forces $C > 0$.

With the above example in mind we now state our main result.

Theorem 1. *Suppose $N \geq 2$.*

1. *Let p, q, N, σ, C be as in Example 1 part 1. Then for sufficiently small C^2 perturbations of the unit ball, say Ω_ε, there exists a positive singular weak solution u of (2) (with $\Omega = \Omega_\varepsilon$) which blows up at exactly one point x_ε (near the origin) and behaves like $u(x) \approx |x - x_\varepsilon|^{-\sigma}$ near x_ε. The proof gives the exact behaviour near x_ε.*

2. *Let p, q, N, σ, C be as in Example 1 part 2. Then for sufficiently small C^2 perturbations of the unit ball, say Ω_ε, there exists a positive weak solution u of (2) (with $\Omega = \Omega_\varepsilon$) with $u \in C^\infty(\Omega_\varepsilon \backslash \{x_\varepsilon\})$ and with $u \in C^{0,\sigma}(\overline{\Omega_\varepsilon})$. In addition u is not in $C^{0,\sigma+\delta}(\overline{\Omega_\varepsilon})$ for any $\delta > 0$.*

1.1 Background

A well studied problem is the existence versus non-existence of positive solutions of the Lane-Emden equation given by

$$\begin{cases} -\Delta u &= u^p \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $1 < p$ and Ω is a bounded domain in $\mathbb{R}^N$ (where $N \geq 3$) with smooth boundary. In the subcritical case $1 < p < \frac{N+2}{N-2}$ the problem is very well understood and $H_0^1(\Omega)$ solutions are classical solutions; see [25]. In the case of $p \geq \frac{N+2}{N-2}$ there are no classical positive solutions in the case of the domain being star-shaped; see [37]. In the case of non star-shaped domains much less is known; see for instance [12,17–19,36]. In the case of $1 < p < \frac{N}{N-2}$ ultra weak solutions (non H_0^1 solutions) can be shown to be classical solutions. For $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ one cannot use elliptic regularity to show ultra weak solutions are classical. In particular in [32] for a general bounded domain in $\mathbb{R}^N$ they construct singular ultra weak solutions with a prescribed singular set. We mention that the weighted Hölder spaces we use in our current work were developed in [32], see also [35].

We now return to (2). The first point is that it is a non variational equation and hence there are various standard tools which are not available anymore. The case $0 < p < 1$ has been studied in [5]. Some relevant monographs for this work include [22, 26, 39]. Many people have studied boundary blow up versions of (2) in the case where $\Delta_p = \Delta_2$ and where one removes the minus sign in front of Δ_p; see for instance [29, 40]. See [1–11, 20, 21, 23, 24, 27, 28, 30, 31, 33, 34, 38] for more results on equations similar to (2). In particular, the interested reader is referred to P.T. Nguyen [33] for recent developments and a bibliography of significant earlier work, where the author studies isolated singularities at 0 of nonnegative solutions of the more general quasilinear equation

$$\Delta u = |x|^\alpha u^p + |x|^\beta |\nabla u|^q \quad in \quad \Omega \setminus \{0\},$$

where $\Omega \subset R^N$ $(N > 2)$ is a C^2 bounded domain containing the origin 0, $\alpha > -2$, $\beta > -1$ and $p, q > 1$, and provides a full classification of positive solutions vanishing on $\partial\Omega$ and the removability of isolated singularities.

1.2 Our approach

Before outlining our approach we mention that our work is motivated by [13–16, 32, 35]. Some of these works deal with a full space or exterior domains; but the linear analysis is still quite similar as compared to what we perform.

We now perform a change of variables to reduce the problem to one on the unit ball; this is take from [16]. Fix $\psi : \overline{B_1} \to \mathbb{R}^N$ be a smooth map and for $\varepsilon > 0$ define

$$\Omega_\varepsilon := \{x + \varepsilon\psi(x) : x \in B_1\}.$$

This domain will be the small perturbation of the unit ball we work on. There is some small $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ one has that Ω_ε is diffeomorphic to the unit ball B_1. Let $y = x + \varepsilon\psi(x)$ for $x \in B_1$ and note there is some $\tilde{\psi}$ smooth such that $x = y + \varepsilon\tilde{\psi}(\varepsilon, y)$ for $y \in \Omega_\varepsilon$. Given $u(y)$ defined on $y \in \Omega_\varepsilon$ or $v(x)$ defined on $x \in B_1$ we define the other via $u(y) = v(x)$. So to find a positive singular solution $u(y)$ of (2) it is sufficient to find a positive singular solution $v(x)$ of some, to be determined equation, on the unit ball. To compute the equation for $v(x)$ we will use the chain rule, but we mentiond that the computation becomes quite involved. We know that

$$u_{y_i} = \sum_{k=1}^{N} v_{x_k} \left(\delta_{ki} + \frac{\partial \tilde{\psi}^k}{\partial y_i} \right) = v_{x_i} + \varepsilon \sum_{k=1}^{N} v_{x_k} \frac{\partial \tilde{\psi}^k}{\partial y_i}.$$

Also a computation shows

$$\begin{aligned} u_{y_i y_j} = \quad & v_{x_i x_j} + \varepsilon \textstyle\sum_{l=1}^N v_{x_i x_l} \tilde{\psi}^l_{y_j} + \varepsilon \sum_{k=1}^N v_{x_k x_j} \tilde{\psi}^k_{y_i} + \varepsilon^2 \sum_{k=1}^N v_{x_k x_j} \tilde{\psi}^j_{y_j} \tilde{\psi}^k_{y_i} \\ & + \varepsilon^2 \textstyle\sum_{k,h=1}^N v_{x_k x_h} \tilde{\psi}^h_{y_j} \tilde{\psi}^k_{y_i} + \varepsilon \sum_{k=1}^N v_{x_k} \tilde{\psi}^k_{y_i y_j}. \end{aligned}$$

In addition

$$u_{y_i}u_{y_j}u_{y_iy_j} = \left(v_{x_i} + \varepsilon \sum_{k=1}^N v_{x_k}\frac{\partial\tilde\psi^k}{\partial y_i}\right)\left(v_{x_j} + \varepsilon \sum_{l=1}^N v_{x_l}\frac{\partial\tilde\psi^l}{\partial y_j}\right)$$

$$\times\left(v_{x_ix_j} + \varepsilon\sum_{l=1}^N v_{x_ix_l}\tilde\psi^l_{y_j} + \varepsilon\sum_{k=1}^N v_{x_kx_j}\tilde\psi^k_{y_i} + \varepsilon^2\sum_{k=1}^N v_{x_kx_j}\tilde\psi^j_{y_j}\tilde\psi^k_{y_i}\right.$$

$$\left.+\varepsilon^2\sum_{k,h=1}^N v_{x_kx_h}\tilde\psi^h_{y_j}\tilde\psi^k_{y_i} + \varepsilon\sum_{k=1}^N v_{x_k}\tilde\psi^k_{y_iy_j}\right)$$

and hence

$$\sum_{ij} u_{y_i}u_{y_j}u_{y_iy_j} = \sum_{ij}\left\{v_{x_i}v_{x_j}v_{x_ix_j}\right.$$

$$+\varepsilon v_{x_i}v_{x_j}\left\{\sum_{l=1}^N v_{x_ix_l}\tilde\psi^l_{y_j} + \sum_{k=1}^N v_{x_kx_j}\tilde\psi^k_{y_i} + \sum_{k=1}^N v_{x_k}\tilde\psi^k_{y_iy_j}\right\}$$

$$+\varepsilon^2 v_{x_i}v_{x_j}\left\{\sum_{k=1}^N v_{x_kx_j}\tilde\psi^j_{y_j}\tilde\psi^k_{y_i} + \sum_{k,h=1}^N v_{x_kx_h}\tilde\psi^h_{y_j}\tilde\psi^k_{y_i}\right\}$$

$$+\varepsilon v_{x_i}v_{x_ix_j}\left\{\sum_{l=1}^N v_{x_l}\tilde\psi^l_{y_j}\right\}$$

$$+\varepsilon^2 v_{x_i}\left\{\sum_{l=1}^N v_{x_ix_l}\tilde\psi^l_{y_j} + \sum_{k=1}^N v_{x_kx_j}\tilde\psi^k_{y_i} + \sum_{k=1}^N v_{x_k}\tilde\psi^k_{y_iy_j}\right\}\left\{\sum_{l=1}^N v_{x_l}\tilde\psi^l_{y_j}\right\}$$

$$+\varepsilon^3 v_{x_i}\left\{\sum_{k=1}^N v_{x_kx_j}\tilde\psi^j_{y_j}\tilde\psi^k_{y_i} + \sum_{k,h=1}^N v_{x_kx_h}\tilde\psi^h_{y_j}\tilde\psi^k_{y_i}\right\}\left\{\sum_{l=1}^N v_{x_l}\tilde\psi^l_{y_j}\right\}$$

$$+\varepsilon v_{x_j}v_{x_ix_j}\left\{\sum_{k=1}^N v_{x_l}\tilde\psi^k_{y_i}\right\}$$

$$+\varepsilon^2 v_{x_j}\left\{\sum_{l=1}^N v_{x_ix_l}\tilde\psi^l_{y_j} + \sum_{k=1}^N v_{x_kx_j}\tilde\psi^k_{y_i} + \sum_{k=1}^N v_{x_k}\tilde\psi^k_{y_iy_j}\right\}\left\{\sum_{k=1}^N v_{x_k}\tilde\psi^k_{y_i}\right\}$$

$$+\varepsilon^3 v_{x_j}\left\{\sum_{k=1}^N v_{x_kx_j}\tilde\psi^j_{y_j}\tilde\psi^k_{y_i} + \sum_{k,h=1}^N v_{x_kx_h}\tilde\psi^h_{y_j}\tilde\psi^k_{y_i}\right\}\left\{\sum_{k=1}^N v_{x_k}\tilde\psi^k_{y_i}\right\}$$

$$+\varepsilon^2 v_{x_ix_j}\left\{\sum_{l=1}^N v_{x_l}\tilde\psi^l_{y_j}\right\}\left\{\sum_{k=1}^N v_{x_k}\tilde\psi^k_{y_i}\right\}$$

$$+\varepsilon^3\left\{\sum_{l=1}^N v_{x_ix_l}\tilde\psi^l_{y_j} + \sum_{k=1}^N v_{x_kx_j}\tilde\psi^k_{y_i} + \sum_{k=1}^N v_{x_k}\tilde\psi^k_{y_iy_j}\right\}\left\{\sum_{l=1}^N v_{x_l}\tilde\psi^l_{y_j}\right\}\left\{\sum_{k=1}^N v_{x_k}\tilde\psi^k_{y_i}\right\}$$

$$\left.+\varepsilon^4\left\{\sum_{k=1}^N v_{x_kx_j}\tilde\psi^j_{y_j}\tilde\psi^k_{y_i} + \sum_{k,h=1}^N v_{x_kx_h}\tilde\psi^h_{y_j}\tilde\psi^k_{y_i}\right\}\left\{\sum_{l=1}^N v_{x_l}\tilde\psi^l_{y_j}\right\}\left\{\sum_{k=1}^N v_{x_k}\tilde\psi^k_{y_i}\right\}\right\}.$$

Now we will partially switch notation back; so we have (and any derivatives of v are understood to be with respect to x)

$$\sum_{i,j=1}^N u_{y_i}u_{y_j}u_{y_iy_j} = \frac{\nabla v\cdot\nabla(|\nabla v|^2)}{2} + \text{ various terms in } \varepsilon,$$

and so we will now simplify the right hand side as

$$\sum_{i,j=1}^{N} u_{y_i} u_{y_j} u_{y_i y_j} = \frac{\nabla v \cdot \nabla(|\nabla v|^2)}{2} + g_0(\varepsilon) \sum_{i,j,k=1}^{N} \left\{ v_{x_i x_j} v_{x_i} v_{x_j} + v_{x_i} v_{x_j} v_{x_k} \right\},$$

where $|g_0(\varepsilon)| \leq C\varepsilon$ for all $|\varepsilon|$ small. We now make some comments on this simplification. Our approach will be to look for solutions of the form $v(x) = w(x) + \phi(x)$ where $w(x) = w(r)$ is the above explicit singular radial solution. We will end up writing out fixed point argument but all these terms that were simplified will not affect the linearized operator; but will only show up in the nonlinear terms. So the exact nature of the terms is not overly important, and in fact if one checks all the dropped terms, they see they are all of the exact for of the two terms we left. Additionally we have dropped the smooth coefficients, but this won't affect anything either.

By [16] we can write $\Delta_y u(y) = \Delta_x v(x) + E_\varepsilon(v)$ where $E_\varepsilon(v)$ is defined by (8). So the equation for v on the unit ball now becomes (after taking into account the prior mentioned simplification)

$$\begin{aligned} 0 &= |\nabla v + \varepsilon A_0 \nabla v|^2 (\Delta v + E_\varepsilon(v)) + \frac{p-2}{2} \nabla v \cdot \nabla(|\nabla v|^2) \\ &\quad + g_0(\varepsilon) \sum \left\{ v_{x_i x_j} v_{x_i} v_{x_j} + v_{x_i} v_{x_j} v_{x_k} \right\} + C|\nabla v + \varepsilon A_0 \nabla v|^{q-p+4} \\ &= (\Delta v)|\nabla v|^2 + \frac{p-2}{2} \nabla v \cdot \nabla(|\nabla v|^2) + C|\nabla v + \varepsilon A_0 \nabla v|^{q-p+4} + H_\varepsilon(v) \end{aligned}$$

where

$$\begin{aligned} H_\varepsilon(v) &:= (\Delta v) 2\varepsilon (A_0 \nabla v) \cdot \nabla v + \varepsilon^2 (\Delta v) |A_0 \nabla v|^2 + E_\varepsilon(v) |\nabla v|^2 \\ &\quad + E_\varepsilon(v)(2\varepsilon A_0 \nabla v) \cdot \nabla v + E_\varepsilon(v) \varepsilon^2 |A_0 \nabla v|^2 \\ &\quad + g_0(\varepsilon) \sum \left\{ v_{x_i x_j} v_{x_i} v_{x_j} + v_{x_i} v_{x_j} v_{x_k} \right\} \end{aligned}$$

and

$$E_\varepsilon(v) := 2\varepsilon \sum_{i,k} v_{x_i x_k} \partial_{y_i} \tilde{\psi}_k + \varepsilon \sum_{i,k} v_{x_k} \partial_{y_i y_i} \tilde{\psi}_k + \varepsilon^2 \sum_{i,j,k} v_{x_j x_k} \partial_{y_i} \tilde{\psi}_j \tilde{\psi}_k. \tag{8}$$

We now hope for small enough ε we can find a solution of the form $v = w + \phi$. If we rewrite the equation putting all the linear in ϕ terms on the left we arrive at

$$\begin{cases} -L(\phi) = \sum_{k=1}^{7} F_k(\phi) + I_\varepsilon(\phi) + H_\varepsilon(w+\phi) & B_1, \\ \phi = 0 & \partial B_1, \end{cases} \tag{9}$$

where

$$F_1(\phi) = \Delta w |\nabla \phi|^2, \quad F_2(\phi) = (\Delta \phi)(2\nabla w \cdot \nabla \phi), \quad F_3(\phi) = (\Delta \phi)|\nabla \phi|^2,$$

$$F_4(\phi) = \frac{p-2}{2} \nabla w \cdot \nabla(|\nabla \phi|^2), \quad F_5(\phi) = (p-2) \nabla \phi \cdot \nabla(\nabla w \cdot \nabla \phi), \quad F_6(\phi) = \frac{(p-2)}{2} \nabla \phi \cdot \nabla |\nabla \phi|^2$$

$$I_\varepsilon(\phi) = C|\nabla w + \nabla \phi + \varepsilon A_0 (\nabla w + \nabla \phi)|^{q-p+4} - C|\nabla w + \nabla \phi|^{q-p+4},$$

$$F_7(\phi) = C \left\{ |\nabla w + \nabla \phi|^{q-p+4} - |\nabla w|^{q-p+4} - (q-p+4)|\nabla w|^{q-p+2} \nabla w \cdot \nabla \phi \right\}.$$

The linear operator L is given by

$$\begin{aligned} L(\phi) \quad := \quad & |\nabla w|^2(\Delta\phi) + (\Delta w)(2\nabla w \cdot \nabla\phi) + (p-2)\nabla w \cdot \nabla(\nabla w \cdot \nabla\phi) \\ & + \frac{(p-2)}{2}\nabla\phi \cdot \nabla|\nabla w|^2 + C(q-p+4)|\nabla w|^{q-p+2}\nabla w \cdot \nabla\phi. \end{aligned}$$

Of crucial importance will be the linear operator L and what functions spaces we work in. Before we consider these issues we want to normalize L by dividing by $|\nabla w|^2$. So instead of considering (9) we will consider

$$\begin{cases} -\tilde{L}(\phi) := \frac{-L(\phi)}{|\nabla w|^2} = \sum_{k=1}^{7} \frac{F_k(\phi)}{|\nabla w|^2} + \frac{I_\varepsilon(\phi)}{|\nabla w|^2} + \frac{H_\varepsilon(w+\phi)}{|\nabla w|^2} & B_1, \\ \phi = 0 & \partial B_1. \end{cases} \tag{10}$$

To obtain a solution of this we will apply the Contraction Mapping Principle to the nonlinear mapping $J_\varepsilon(\phi) = \psi$ (for $\phi \in X$ where X is yet to be determined and of course this mapping is not well defined yet)

$$\begin{cases} -\tilde{L}(\psi) = \sum_{k=1}^{7} \frac{F_k(\phi)}{|\nabla w|^2} + \frac{I_\varepsilon(\phi)}{|\nabla w|^2} + \frac{H_\varepsilon(w+\phi)}{|\nabla w|^2} & B_1, \\ \psi = 0 & \partial B_1. \end{cases} \tag{11}$$

The exact form of $\tilde{L}$ will be crucial for us. A computation shows that we can write

$$\tilde{L}(\phi) = \Delta\phi + \gamma\phi_{rr} + \frac{\alpha\phi_r}{r},$$

where $\gamma := p-2$ and

$$\alpha := 2(N-1) - 2(p-1)(\sigma+1) - C(q-p+4)\sigma^{q-p+1}, \tag{12}$$

where C is given by (5).

We will examine this operator in Section 2.

2 Linear theory

We study the linear theory for the problem in two different cases (1) The singular case and (2) The Hölder continues case in the following subsections.

2.1 The singular case

We first define the function spaces. For $0 < s \leq \frac{1}{2}$ define $A_s := \{x \in \mathbb{R}^N : s < |x| < 2s\}$ and for $\sigma \in \mathbb{R}$ and $N < t < \infty$ define the spaces $Y = Y_{t,\sigma}$ and $X = X_{t,\sigma}$ with norms given by

$$\|f\|_Y^t := \sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N} \int_{A_s} |f(x)|^t dx \text{ and}$$

$$\|\phi\|_X^t := \sup_{0<s\leq\frac{1}{2}} s^{\sigma t-N} \left\{ \int_{A_s} |\phi|^t dx + s^t \int_{A_s} |\nabla\phi|^t dx + s^{2t} \int_{A_s} |D^2\phi|^t dx \right\},$$

where for the space X we impose the boundary condition $\phi = 0$ on ∂B_1. We now define the closed subspecies of X and Y respectively X_1, Y_1 where we remove the first mode. So to define this properly we need to introduce the spherical harmonics.

Consider the Laplace-Beltrami operator $\Delta_{S^{N-1}} = \Delta_\theta$ on S^{N-1} and the eigenpairs

$$-\Delta_\theta \psi_k(\theta) = \lambda_k \psi_k(\theta), \quad \theta \in S^{N-1},$$

and note that $\lambda_0 = 0, \psi_0 = 1$ (multiplicity 1); $\lambda_1 = N-1$ with multiplicity N and $\lambda_2 = 2N$. Given $\phi \in X, f \in Y$ we write

$$\phi(x) = \sum_{k=0}^{\infty} a_k(r)\psi_k(\theta), \qquad f(x) = \sum_{k=0}^{\infty} b_k(r)\psi_k(\theta),$$

and so we define

$$X_1 := \left\{ \phi \in X : \phi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta) \right\}, \qquad \text{note there is no } k=0 \text{ mode}$$

and anagolous for Y. Note we are abusing notation by not showing the correct multiplicity for modes which have multiplicity greater than one; but this isn't an issue for the procedures we perform. For $\gamma, \alpha \in \mathbb{R}$ we define the operator

$$L(\phi)(x) := L_{\gamma,\alpha}(\phi)(x) := \Delta\phi(x) + \gamma\phi_{rr}(x) + \frac{\alpha}{r}\phi_r(x).$$

Note we can write the operator as

$$L_{\gamma,\alpha}(\phi)(x) = \Delta\phi + \gamma \sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2}\phi_{x_i x_j} + \frac{\alpha}{|x|^2} x \cdot \nabla\phi(x).$$

In this section we will prove various results regarding this operator $L = L_{\gamma,\alpha}$. For explicit values of γ, α this operator $L_{\gamma,\alpha}$ will be exactly the operator $\tilde{L}$ from the previous section. In this section the values of γ, α, σ will satisfy a few constraints but are otherwise arbitrary. Of course when we apply the results of this section to the explicit linear operator $L = \tilde{L}$ we have exact values of these parameters in mind. So we state these assumptions now.

Values of parameters. We take $1 < p < N$, $\frac{(p-1)N}{N-1} < q < p$

$$\begin{aligned} &\gamma := p-2, \\ &\sigma := \tfrac{p-q}{q-p+1}, \\ &\alpha := 2(N-1) - 2(p-1)(\sigma+1) - C(q-p+4)\sigma^{q-p+1}, \quad \text{then a computation shows} \\ &N-2-\gamma+\alpha = \tfrac{p-1}{q-p+1} - (N-1)(q-p+1), \\ &\beta_1^- = \tfrac{-1}{q-p+1}, \\ &\sigma + \beta_1^- = -1. \end{aligned} \tag{13}$$

For the definition of $\beta_k^\pm$ see the proof of the following lemma. A computation shows that $N-2-\gamma+\alpha$ changes sign in the interval $\frac{(p-1)N}{N-1} < q < p$.

As we have already mentioned we want the various parameters to satisfy the above requirements. But for various parts of the linear theory we can drop various assumptions.

Lemma 1. *Suppose $-1 < \gamma < N-2$ and $0 < \sigma < -\beta_1^-$. Suppose $\phi \in X_1$ is such that $L(\phi) = 0$ in $B_1 \backslash \{0\}$. Then $\phi = 0$.*

Proof. We write $\phi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$ and note

$$(1+\gamma)a_k''(r) + \frac{(N-1+\alpha)}{r}a_k'(r) - \frac{\lambda_k a_k(r)}{r^2} = 0 \quad 0 < r < 1$$

with $a_k(1) = 0$. Then we have $a_k(r) = C_k(r^{\beta_k^+} - r^{\beta_k^-})$ after considering the boundary condition, where

$$\beta_k^{\pm} := -\frac{(N-2-\gamma+\alpha)}{2(1+\gamma)} \pm \frac{\sqrt{(N-2-\gamma+\alpha)^2 + 4(1+\gamma)\lambda_k}}{2(1+\gamma)}.$$

Note that if $\sigma < -\beta_k^-$ then to have $\phi \in X$ we must have $C_k = 0$. Hence we see the kernel is empty provided $\sigma < -\beta_1^-$ after taking into account the monotonicity in k. □

Lemma 2. *(Onto ode estimates; $k \geq 1$) Suppose $-1 < \gamma < N-2$ and $0 < \sigma < -\beta_1^-$. Then for each $k \geq 1$ there is some C_k such that for each b_k there is some a_k such that $L(a_k\psi_k) = b_k\psi_k$ and $\|a_k\psi_k\|_X \leq C_k\|b_k\psi_k\|_Y$ and $a_k(1) = 0$.*

Proof. We now prove the desired onto estimate for each mode $k \geq 1$. For each $k \geq 0$ consider

$$(1+\gamma)a_k''(r) + \frac{(N-1+\alpha)a_k'(r)}{r} - \frac{\lambda_k a_k(r)}{r^2} = b_k(r) \quad 0 < r < 1 \tag{14}$$

with $a_k(1) = 0$. Using the variation of parameters method we obtain solutions of the form

$$(\gamma+1)(\beta_k^- - \beta_k^+)a_k(r) = r^{\beta_k^-}\int_{T_2}^{r} \frac{b_k(\tau)}{\tau^{\beta_k^- - 1}}d\tau - r^{\beta_k^+}\int_{T_1}^{r} \frac{b_k(\tau)}{\tau^{\beta_k^+ - 1}}d\tau + C_k r^{\beta_k^+} + D_k r^{\beta_k^-},$$

where C_k, D_k are free parameters and we are free to choose T_i suitably; we need to pick these parameters such that we get the desired estimate on a_k and such that $a_k(1) = 0$. We will choose $T_2 = 0$, $T_1 = 1$, $D_k = 0$ and we leave C_k free for now and hence we get

$$(p-1)(\beta_k^- - \beta_k^+)a_k(r) = r^{\beta_k^-}\int_0^r \frac{b_k(\tau)}{\tau^{\beta_k^- - 1}}d\tau - r^{\beta_k^+}\int_1^r \frac{b_k(\tau)}{\tau^{\beta_k^+ - 1}}d\tau + C_k r^{\beta_k^+},$$

and note this is an acceptable choice of T_2 provided $\frac{b_k(t)}{t^{\beta_k^- - 1}} \in L^1(0,1)$, which we assume for now. For simplicity we normalize $\|b_k\psi_k\|_Y \leq 1$ and hence there is some $\tilde{C}_k$ such that

$$\int_s^{2s} |b_k(\tau)|^t d\tau \leq \tilde{C}_k s^{1-t(2+\sigma)} \quad 0 < s \leq \frac{1}{2}. \tag{15}$$

We now prove that $\frac{b_k(\tau)}{\tau^{\beta_k^- - 1}} \in L^1(0,1)$. To see this note that $\beta_k^- = -\sigma - \varepsilon_k$ where $\varepsilon_k > 0$.

$$\begin{aligned}
\int_0^1 \frac{|b_k(\tau)|}{\tau^{\beta_k^- - 1}}d\tau &\leq \sum_{i=0}^{\infty}\int_{2^{-i-1}}^{2^{-i}} |b_k(\tau)|\tau^{\sigma+1+\varepsilon_k}d\tau \\
&\leq C\sum_{i=0}^{\infty}\left(\frac{1}{2^i}\right)^{\sigma+1+\varepsilon_k+\frac{1}{t'}}\left(\int_{2^{-i-1}}^{2^{-i}} |b_k(\tau)|^t d\tau\right)^{\frac{1}{t}}
\end{aligned}$$

and using the above estimate on b_k with $s = 2^{-i-1}$ gives a result like

$$\int_0^1 \frac{|b_k(\tau)|}{\tau^{\beta_k^- -1}} d\tau \le \hat{C}_k \sum_{i=0}^{\infty} \frac{1}{2^{iB}},$$

where $B > 0$ exactly when $\beta_1^- < -\sigma$.

We first examine the term given by

$$r^{\beta_k^-} \int_0^r \frac{b_k(\tau)}{\tau^{\beta_k^- -1}} d\tau + C_k r^{\beta_k^+}$$

and we choose

$$C_k := -\int_0^1 \frac{b_k(\tau)}{\tau^{\beta_k^- -1}} d\tau.$$

Note with this choice of C_k we have the needed zero boundary condition for this term (and its clear the other term has the needed boundary condition) hence $a_k(1) = 0$. We now get the estimate. Firstly note by the previous argument to show the needed integrand is $L^1(0,1)$ we have $|C_k|$ is bounded by a constant depending just on k and hence its clear that $\|C_k r^{\beta_k^+} \psi_k\|_X$ is bounded by a constant just depending on k. We now consider the integral term.

A computation shows

$$\begin{aligned}
\int_0^r \frac{|b_k(\tau)|}{\tau^{\beta_k^- -1}} d\tau \;&\le C_k \;\; \sum_{i=0}^{\infty} (r2^{-i})^{1-\beta_k^-} \int_{r2^{-i-1}}^{r2^{-i}} |b_k(\tau)| d\tau \\
&\le \;\; C_k \tilde{C}_k \sum_{i=0}^{\infty} (r2^{-i})^{1-\beta_k^- + \frac{1}{t'}} \left(\int_{r2^{-i-1}}^{r2^{-i}} |b_k(\tau)|^t d\tau \right)^{\frac{1}{t}} \\
&\le \;\; C_k \tilde{C}_k \sum_{i=0}^{\infty} (r2^{-i})^{1-\beta_k^- + \frac{1}{t'}} \\
&\le \;\; C_{k,1} \sum_{i=0}^{\infty} (r2^{-i})^{1-\beta_k^- + \frac{1}{t'} + \frac{1}{t} - 2 - \sigma} \\
&= \;\; r^{-\beta_k^- - \sigma} C_{k,1} \sum_{i=0}^{\infty} \frac{1}{(2^{-\beta_k^- - \sigma})^i}
\end{aligned}$$

and since $-\beta_k^- - \sigma > 0$ we see we get the estimate

$$\sup_{0<r<1} r^{\beta_k^- + \sigma} \int_0^r \frac{|b_k(\tau)|}{\tau^{\beta_k^- -1}} d\tau \le \hat{C}_k,$$

and from this one can show that

$$\left\| \psi_k r^{\beta_k^-} \int_0^r \frac{b_k(\tau)}{\tau^{\beta_k^- -1}} d\tau \right\|_X \le C_{k,2}.$$

We now examine the term given by

$$r^{\beta_k^+} \int_1^r \frac{b_k(\tau)}{\tau^{\beta_k^+ -1}} d\tau =: r^{\beta_k^+} g_k(r).$$

Note that we can write (for integers $n \geq 1$)

$$g_k(2^{-n}) = \sum_{i=1}^{n} (g_k(2^{-i}) - g_k(2^{-i+1})) \quad \text{and hence} \quad |g_k(2^{-n})| \leq \sum_{i=1}^{n} |g_k(2^{-i}) - g_k(2^{-i+1})|.$$

A computation similar to the previous one shows

$$\begin{aligned} |g_k(2^{-n})| &\leq \sum_{i=1}^{n} \int_{2^{-i}}^{2^{1-i}} \frac{|b_k(\tau)|}{\tau^{\beta_k^+ - 1}} d\tau \\ &\leq C_k \sum_{i=1}^{n} 2^{i(\beta_k^+ + \sigma)} \\ &\leq D_k \left(1 + 2^{n(\beta_k^+ + \sigma)}\right), \end{aligned}$$

and from this we see

$$(2^{-n})^{\beta_k^+ + \sigma} |g_k(2^{-n})| \leq \tilde{D}_k$$

for all $n \geq 1$. This gives us the desired zero order estimate at least for the values of $r \in \{2^{-n} : n \geq 1 \text{ an integer}\}$. One can extend the above estimate for all values of r and hence combining all the above results gives us the needed zero order estimate on $a_k(r)$. The higher order portions of the norm of a_k can be obtained from the zero order estimates after consider the equation that a_k satisfies.

□

The following are some standard local estimates, at least in the case of $L_{\gamma,0}$ replaced with Δ.

Lemma 3. *Let $\gamma > -1$ and $1 < t < \infty$. Then there is some $C > 0$ such that*

$$\|\phi\|_{W^{2,t}(1<|x|<2)} \leq C \|L_{\gamma,0}(\phi)\|_{L^t(\frac{1}{2}<|x|<4)} + C\|\phi\|_{L^t(\frac{1}{2}<|x|<4)},$$

for all sufficiently smooth ϕ. Then there is some $C > 0$ such that

$$\|\phi\|_{W^{2,t}(1<|x|<2)} \leq C \|L_{\gamma,0}(\phi)\|_{L^t(\frac{1}{2}<|x|<2)} + C\|\phi\|_{L^t(\frac{1}{2}<|x|<2)},$$

for all sufficiently smooth ϕ with $\phi = 0$ on $|x| = 2$.

Proof. Note when $\gamma > -1$ that $L_{\gamma,0}$ is uniformly elliptic and the coefficients are smooth away from the origin. So the proof of the above results follow exactly as in the case of $L_{\gamma,0}$ replaced with Δ. □

Corollary 1. *Let $\gamma > -1$, $\alpha \in \mathbb{R}$ and $1 < t < \infty$. Then there is some $C > 0$ such that*

$$\|\phi\|_{W^{2,t}(1<|x|<2)} \leq C \|L_{\gamma,\alpha}(\phi)\|_{L^t(\frac{1}{2}<|x|<4)} + C\|\phi\|_{L^t(\frac{1}{2}<|x|<4)} + C\|\nabla\phi\|_{L^t(\frac{1}{2}<|x|<4)} \tag{16}$$

for all sufficiently smooth ϕ. Then there is some $C > 0$ such that

$$\|\phi\|_{W^{2,t}(1<|x|<2)} \leq C \|L_{\gamma,\alpha}(\phi)\|_{L^t(\frac{1}{2}<|x|<2)} + C\|\phi\|_{L^t(\frac{1}{2}<|x|<2)} + C\|\nabla\phi\|_{L^t(\frac{1}{2}<|x|<2)}, \tag{17}$$

for all sufficiently smooth ϕ with $\phi = 0$ on $|x| = 2$.

Proof. The result follows by writing $L_{\gamma,\alpha}(\phi) = L_{\gamma,0}(\phi) + \frac{\alpha x\cdot\nabla\phi}{|x|^2}$ and using the previous result. $\square$

Theorem 2. *Suppose* $-1 < \gamma < N-2$ *and* $0 < \sigma < -\beta_1^-$. *Then there is some* $C > 0$ *such that for all* $f \in Y_1$ *there is some* $\phi \in X_1$ *such that* $L_{\gamma,\alpha}(\phi) = f$ *in* $B_1\backslash\{0\}$ *and* $\|\phi\|_X \le C\|f\|_Y$.

Proof. A standard argument along with Lemma 2 shows that for all $m \ge 1$ there is some C_m such for all $f(x) = \sum_{k=1}^m b_k(r)\psi_k(\theta)$ there is some $\phi(x) = \sum_{k=1}^m a_k(r)\psi_k(\theta)$ (with $a_k(1) = 0$) such that $L(\phi) = f$ in $B_1\backslash\{0\}$ and $\|\phi\|_X \le C_m\|f\|_Y$. So by a density argument it is sufficient to show that C_m is bounded. Suppose not, then there is some $\phi_m \in X_1$ (finite number of nonzero modes) and $f_m \in Y_1$ such that $L(\phi_m) = f_m$ and $\|f_m\|_Y \to 0$, $\|\phi_m\|_X = 1$.

Claim 1. We claim that $\sup_{0<s\le\frac12}\left\{s^{\sigma t-N}\int_{A_s}|\phi_m|^t dx + s^{(\sigma+1)t-N}\int_{A_s}|\nabla\phi_m|^t dx\right\} = \sup_{0<s\le\frac12}\Phi_m(s) \to 0$; so towards a contradiction we assume this quantity is greater or equal $4\varepsilon_0 > 0$ for all m. So there is some $0 < s_m \le \frac12$ such that $\Phi_m(s_m) \ge 2\varepsilon_0$.
Case (i). s_m bounded away from zero. **Case (ii).** $s_m \to 0$.

Case (i). Using an argument as in the proof of (17) we see that for all $0 < s < 1$ we have ϕ_m bounded in $W^{2,t}(s < |x| < 1)$. Hence by a diagonal argument we can pass to a subsequance to find some ϕ such that $\phi_m \rightharpoonup \phi$ in $W^{2,t}_{loc}(\overline{B_1}\backslash\{0\})$. Also note that $f_m \to 0$ in $L^t_{loc}(\overline{B_1}\backslash\{0\})$ and hence ϕ satisfies $L_{\gamma,\alpha}(\phi) = 0$ in $B_1\backslash\{0\}$ with $\phi = 0$ on ∂B_1. Since s_m bounded away from zero we can use the noted convergence to see ϕ is nonzero. Hence if we can show that $\phi \in X_1$ then we'd obtain the desired contradiction after recalling the kernel is empty. Fix $0 < s \le \frac12$ and note by the stated weak convergence and weak lower semi continuity of L^p norms we see $\phi \in X_1$.

Case (ii). Define $\zeta_m(x) := s_m^\sigma\phi_m(s_m x)$ for $0 < |x| < \frac{1}{s_m}$. Note that since ϕ_m has no $k=0$ mode ζ_m also have no $k=0$ mode. A computation shows that

$$L_{\gamma,\alpha}(\zeta_m) = g_m(x) := s_m^{2+\sigma} f_m(s_m x) \quad 0 < |x| < \frac{1}{s_m}, \tag{18}$$

with $\zeta_m = 0$ on $|x| = \frac{1}{s_m}$. For k a large integer we set $E_k := \{x \in \mathbb{R}^N : \frac1k < |x| < k\}$ and $\tilde{E}_k := \{x \in \mathbb{R}^N : \frac{1}{2k} < |x| < 2k\}$. By the local estimates there is some C_k such that

$$\|\zeta_m\|_{W^{2,t}(E_k)} \le C_k\left\{\|g_m\|_{L^t(\tilde{E}_k)} + \|\zeta_m\|_{L^t(\tilde{E}_k)} + \|\nabla\zeta_m\|_{L^t(\tilde{E}_k)}\right\} \le \tilde{C}_k,$$

and hence by a diagonal argument we see there is some ζ such that $\zeta_m \rightharpoonup \zeta$ in $W^{2,t}_{loc}(\mathbb{R}^N\backslash\{0\})$ and hence we have $L_{\gamma,\alpha}(\zeta) = 0$ in $\mathbb{R}^N\backslash\{0\}$. Also note that a computation shows that

$$\int_{1<|x|<2}\left\{|\zeta(x)|^t + |\nabla\zeta(x)|^t dx\right\} \ge 2\varepsilon_0,$$

and hence $\zeta \ne 0$. We write $\zeta(x) = \sum_{k=1}^\infty a_k(r)\psi_k(\theta)$ and as usual we have

$$(\gamma+1)a_k''(r) + \frac{(N-1+\alpha)a_k'(r)}{r} - \frac{\lambda_k a_k(r)}{r^2} = 0, \qquad 0 < r < \infty,$$

and hence, as before, we have $a_k(r) = C_k r^{\beta_k^+} + D_k r^{\beta_k^-}$. To get a contradiction we hope to show that $C_k = D_k = 0$, but to do this we need some estimates on ζ near $r = 0$ and $r = \infty$. For $i \in \mathbb{R}$ a computation shows that

$$\int_{2^i<|x|<2^{i+1}} |\zeta_m(x)|^t dx \le (2^i)^{N-\sigma t}, \qquad \int_{2^i<|x|<2^{i+1}} |\nabla \zeta_m(x)|^t dx \le (2^i)^{N-(\sigma+1)t},$$

and we can pass to the limit in these estimates. Using the first estimate we see there is some $\tilde{C}_k > 0$ such that

$$\int_{2^i}^{2^{i+1}} |a_k(r)|^t r^{N-1} dr \le \tilde{C}_k 2^{i(N-\sigma t)},$$

for all $i \in \mathbb{R}$. Using a change of variables this gives

$$\int_1^2 s^{N-1} \left| C_k s^{\beta_k^+} (2^i)^{\beta_k^+ + \sigma} + D_k s^{\beta_k^-} (2^i)^{\beta_k^- + \sigma} \right|^t ds \le \tilde{C}_k$$

for all $i \in \mathbb{R}$ and note for all k we have $\beta_k^+ \neq \beta_k^-$. Hence the only way we can possibly have one of C_k or D_k nonzero is that either we have $\beta_k^+ + \sigma = 0$ or $\beta_k^- + \sigma = 0$. We now recall the assumptions on σ and using the monotonicity in k of $\beta_k^\pm$ we have the desired result. □

For the following lemma we use the exact values of the parameters.

Lemma 4. *(Onto estimate for $k = 0$ mode) Suppose the parameters satisfy (13) and set $\beta := \frac{N-1+\alpha}{1+\gamma}$ (which implies $\beta - \sigma - 1 < 0$). There is some $C_0 > 0$ such that for all b_0 there is some a_0 which satisfies (14) for $k = 0$ and $\|a_0\|_X \le C_0 \|b_0\|_Y$.*

Proof. We look for a solution $a_0(r)$ of

$$(1+\gamma) a_0''(r) + \frac{(N-1+\alpha)}{r} a_0'(r) = b_0(r), \quad 0 < r < 1$$

with $a_0(1) = 0$. We normalize b_0 such that its Y norm is 1. Note we can use the integrating factor method to get an explicit formula for the solution. If one does this (and taking $a_0'(1) = 0$) we arrive at

$$a_0(R) = \int_R^1 \left\{ \frac{1}{r^\beta} \int_r^1 \frac{\tau^\beta b_0(\tau)}{1+\gamma} d\tau \right\} dr.$$

We now get the needed estimate on a_0 but instead we get an estimate for a_0'. So we have

$$(1+\gamma) r^\beta |a_0'(r)| \le \int_r^1 \tau^\beta |b_0(\tau)| d\tau.$$

Taking $r = \frac{1}{2^n}$ for n a large integer, we have

$$\begin{aligned} \left(\frac{1}{2^n}\right)^\beta \left|a_0'\left(\frac{1}{2^n}\right)\right| &\le \frac{1}{\gamma+1} \sum_{i=1}^n \int_{2^{-i}}^{2^{1-i}} \tau^\beta |b_0(\tau)| d\tau \\ &\le C \sum_{i=1}^n \frac{1}{2^{i\beta + \frac{i}{t'}}} \left(\int_{2^{-i}}^{2^{1-i}} |b_0(\tau)|^t d\tau \right)^{\frac{1}{t}} \qquad C \text{ independent of } n \\ &\le C \sum_{i=1}^n \frac{1}{2^{i(\beta + \frac{1}{t'} + \frac{1}{t} - 2 - \sigma)}}, \qquad \text{after using (15)} \\ &\le C_1 \left(1 + 2^{(1+\sigma-\beta)n}\right) \end{aligned}$$

after using the fact this is a geometric series. Rearranging this we arrive at

$$(2^{-n})^{\sigma+1}|a_0'(2^{-n})| \le C_1 \left(1 + 2^{n(\beta-\sigma-1)}\right),$$

and recall $\beta - \sigma - 1 < 0$. From this we see the right hand side is bounded independently of n. This shows that for $r = 2^{-n}$ we have the estimate $r^{\sigma+1}|a_0'(r)| \le C$. Standard arguments extend the result to the other values of r. This will give us the needed estimates to bound the zero and first order terms in the X norm of a_0. To get bounds on the zero order terms one integrates these first order estimates; to get the second order terms we use the ode directly; we omit the details.

□

Corollary 2. *Suppose the parameters satisfy (13). Then there is some $C > 0$ such that for all $f \in Y$ there is some $\phi \in X$ which satisfies $L_{\gamma,\alpha}(\phi) = f$ in $B_1 \backslash \{0\}$ with $\phi = 0$ on ∂B_1. Moreover one has $\|\phi\|_X \le C\|f\|_Y$.*

Proof. Given $f \in Y$ we write $f = f_0 + f_1$ where $f_1 \in Y_1$ and $f_0 = f_0(r)$. We claim there is some $C_1 > 0$ such that $\|f_0\|_Y \le C_1\|f\|_Y$ independent of f. To see this we use the fact that $f_0(r) = C_N \int_{|\theta|=1} f(r\theta)d\theta$ where C_N depends on N. We then write out the Y norm of f_0 and apply Jensen's inequality to arrive at the desired result. From this we get the same estimate for f_1 but with a larger C_1 if need be. Now let $f \in Y$ and decompose as above and we let $\phi = \phi_0 + \phi_1$ where $L(\phi_i) = f_i$ in $B_1 \backslash \{0\}$ with $\phi_i = 0$ on ∂B_1. Then if let C_4 denote the maximum of the C's from Theorem 2 and Lemma 4. Then we have

$$\|\phi\|_X \le \|\phi_0\|_X + \|\phi_1\|_X \le C_4\|f_0\|_Y + C_4\|f_1\|_Y \le 2C_4C_1\|f\|_Y,$$

which gives us the desired estimate. □

2.2 The Hölder continuous case

Here we examine the needed linear theory to linearize around the radial Hölder continuous solution from Example 1 Case 2 where $w(r) = 1 - r^\sigma$ where $\sigma := \frac{q-p}{q-p+1}$. If one takes the same approach as in the singular case, they see we need to examine the operator $L = L_{\gamma,\alpha}$ where

$$L(\phi)(x) := L_{\gamma,\alpha}(\phi)(x) := \Delta\phi(x) + \gamma\phi_{rr}(x) + \frac{\alpha}{r}\phi_r(x)$$

and

$$\begin{aligned} &\gamma := p - 2 \\ &\sigma := \tfrac{q-p}{q-p+1}, \\ &\alpha := 2(N-1) + 2(p-1)(\sigma-1) - C(q-p+4)\sigma^{q-p+1}, \qquad C \text{ defined in (6).} \end{aligned}$$

The spaces we work in are the same as before (again we have $N < t < \infty$) except now note the change of sign in front of σ; define the spaces $Y = Y_{t,\sigma}$ and $X = X_{t,\sigma}$ with norms given by

$$\|f\|_Y^t := \sup_{0<s\le\frac{1}{2}} s^{(2-\sigma)t-N} \int_{A_s} |f(x)|^t dx \text{ and}$$

$$\|\phi\|_X^t := \sup_{0<s\le\frac{1}{2}} s^{-\sigma t-N} \left\{ \int_{A_s} |\phi|^t dx + s^t \int_{A_s} |\nabla\phi|^t dx + s^{2t} \int_{A_s} |D^2\phi|^t dx \right\}$$

where for the space X we impose the boundary condition $\phi = 0$ on ∂B_1. We now define the closed subspecies of X and Y, respectively X_1, Y_1 where we remove the $k = 0$ mode.

Lemma 5. *Define α, γ, σ as above and define $\beta_k^{\pm}$ as before and we set $\beta := \frac{N-1+\alpha}{\gamma+1}$. Then*

$$\beta < 1-\sigma, \qquad \beta_k^- < -\sigma, \qquad \beta_k^+ > \sigma,$$

for all $k \geq 1$.

Proof. Note that $\beta - 1 = \frac{N-2-\gamma+\alpha}{\gamma+1}$ or $\beta - 1 = \frac{1}{q-p+1} - \frac{(N-1)(q-p+1)}{p-1}$ and finally

$$\beta - 1 + \sigma = 1 - \frac{(N-1)(q-p+1)}{p-1} = \frac{(p-1)-(N-1)(q-p+1)}{p-1} < 0$$

since $q > \frac{N(p-1)}{N-1}$.
Also Note that $\beta_1^- + \sigma = -1 < 0$. Thus $\beta_1^- < -\sigma$ and since $\beta_k^- < \beta_1^-$, so we proved the claim. Finally, notice that $\beta_1^+ - \sigma > 0$

$$\begin{aligned}
\beta_1^- - \sigma = & -\tfrac{(N-2-\gamma+\alpha)}{2(1+\gamma)} + \tfrac{\sqrt{(N-2-\gamma+\alpha)^2+4(1+\gamma)(N-1)}}{2(1+\gamma)} - \sigma \\
= & -\tfrac{\frac{p-1}{q-p+1}-(N-1)(q-p+1)}{2(p-1)} + \tfrac{(N-1)(q-p+1)+\frac{p-1}{q-p+1}}{2(p-1)} - \sigma \\
= & \tfrac{(N-1)(q-p+1)}{(p-1)} - \tfrac{q-p}{q-p+1} \\
\geq & \ 1 - \tfrac{q-p}{q-p+1} = \tfrac{1}{q-p+1} > 0.
\end{aligned}$$

This implies that $\beta_k^+ > \sigma$. ☐

As in the previous section, our goal is to develop a linear theory to consider

$$L(\phi) = L_{\gamma,\alpha}(\phi) = f \quad \text{in } B_1 \backslash \{0\}, \qquad \phi = 0 \quad \text{on } \partial B_1. \tag{19}$$

As before we use spherical harmonics

$$f(x) = \sum_{k=0}^{\infty} b_k(r)\psi_k(\theta), \quad \phi(x) = \sum_{k=0}^{\infty} a_k(r)\psi_k(\theta),$$

and then we need $a_k(r)$ to satisfy

$$(1+\gamma)a_k''(r) + \frac{(N-1+\alpha)}{r}a_k'(r) - \frac{\lambda_k a_k(r)}{r^2} = b_k(r) \quad 0 < r < 1, \tag{20}$$

with $a_k(1) = 0$. As before we separate the $k = 0$ mode.

Lemma 6. *($k = 0$ mode). There is some $C_0 > 0$ such that for all $b_0(r)$ there is some $a_0(r)$ which satisfies (20) and $\|a_0\|_X \leq C\|b_0\|_Y$.*

Proof. For $0 < R \leq 1$ define

$$a_0(R) := \int_R^1 \left(\frac{1}{r^\beta} \int_r^1 \frac{b_0(\tau)\tau^\beta}{\gamma+1} d\tau - \frac{T}{r^\beta} \right) dr,$$

where we define T such that

$$T\int_0^1 r^{-\beta}dr = \int_0^1 \left(\frac{1}{r^\beta}\int_r^1 \frac{b_0(\tau)\tau^\beta}{\gamma+1}d\tau \right) dr.$$

Note that $\beta < 1$ and hence the integrals over $(0,1)$ not involving b_0 are finite. We normalize b_0 via $\|b_0\|_Y \le 1$. For the time being we adjust the Y norm via $\|b_0\|_{\hat{Y}} := \sup_{0<|x|<1}|x|^{2-\sigma}|b_0(x)|$ (and we again normalize b_0) we then easily see that $|T| \le \tilde{C}$. Also note that

$$-a_0'(R) = \frac{1}{R^\beta}\int_R^1 \frac{b_0(\tau)\tau^\beta}{\gamma+1}d\tau - \frac{T}{R^\beta},$$

and then note

$$|a_0'(R)| \le \frac{\tilde{C}}{R^\beta} + \frac{D}{R^\beta}\int_R^1 \tau^{\beta-2+\sigma}d\tau,$$

which is bounded above by $\tilde{C}R^{-\beta} + D_2R^{-1+\sigma} \le D_3R^{-1+\sigma}$ and hence we obtain the estimate $R^{1-\sigma}|a_0'(R)| \le D_4$ for all $0 < R < 1$. Let $0 < R < 1$ and take $0 < \varepsilon < R$ and note we have

$$|a_0(R) - a_0(\varepsilon)| \le \int_\varepsilon^R |a_0'(r)|dr \le \int_\varepsilon^R D_4 r^{-1+\sigma}dr \le D_5(R^\sigma - \varepsilon^\sigma),$$

and if we can show a_0 is continuous at $r = 0$ with $a_0(0) = 0$ then by sending $\varepsilon \searrow 0$ we'd have $|a_0(R)| \le D_5R^\sigma$. Note we chose T exactly such that $a_0(0) = 0$ and one easily sees the needed continuity.

Of course since our space is Y and not $\hat{Y}$ the above doesn't show anything. One needs to use Hölder's inequality argument coupled with the dyadic intervals as we used before to show we can replace the $\hat{Y}$ norm with the Y norm and obtain the same estimate. The second order estimates on a_0 come directly from the ode; we omit the details. □

We now consider obtaining the need estimates on the higher modes. As before we start with the solution of (20) given by

$$(p-1)(\beta_k^- - \beta_k^+)a_k(r) = r^{\beta_k^-}\int_0^r \frac{b_k(\tau)}{\tau^{\beta_k^- -1}}d\tau - r^{\beta_k^+}\int_1^r \frac{b_k(\tau)}{\tau^{\beta_k^+ -1}}d\tau + C_k r^{\beta_k^+}.$$

We choose C_k such that $a_k(1) = 0$. Note the condition that $\beta_k^+ > \sigma$ implies that that $r^{\beta_k^+} \in X$ (except for the fact it doesn't satisfy the needed boundary condition). As in the proof of the estimate for the $k = 0$ mode, if we replace the Y norm with the $\hat{Y}$ norm we see the conditions from Lemma 5 are sufficient to show that

$$\sup_{0<r<1}\left(r^{1-\sigma}|a_k'(r)| + r^{-\sigma}|a_k(r)|\right) \le C_k \sup_{0<r<1} r^{2-\sigma}|b_k(r)|.$$

As before one needs to replace the $\hat{Y}$ norm with the Y norm and use some additional arguments to obtain the desired result.

Finally one can argue as in the previous section and combine the modes to obtain the following theorem.

Theorem 3. *Let $N \ge 2$, $p > 1$ and $q > \max\left\{p, \frac{N(p-1)}{N-1}\right\}$ and σ, γ, α be as above. Then there is some $C > 0$ such that for all $f \in Y$ there is some $\phi \in X$ which satisfies (19) and $\|\phi\|_X \le C\|f\|_Y$.*

3 Fixed point theory

We consider the singular and and Hölder continuous case separately in the following sections.

3.1 The singular case

Here, we know that $\sigma = \frac{p-q}{q-p+1}$, $p-1 \le q \le p$ and the norms given by

$$\|f\|_Y^t := \sup_{0<s\le\frac{1}{2}} s^{(2+\sigma)t-N} \int_{A_s} |f(x)|^t dx$$

and

$$\|\phi\|_X^t := \sup_{0<s\le\frac{1}{2}} s^{\sigma t-N} \left\{ \int_{A_s} |\phi|^t dx + s^t \int_{A_s} |\nabla\phi|^t dx + s^{2t} \int_{A_s} |D^2\phi|^t dx \right\}.$$

Also a computation shows that

$$\frac{\Delta w}{|\nabla w|^2} = \sigma(\sigma+2-N)r^\sigma.$$

Recall we have defined $J_\varepsilon(\phi) = \psi$, where ψ satisfies (11). In order to obtain a solution ϕ of (9) we will show that J_ε is a contraction on B_r where B_r is the closed ball of radius r centered at the origin in X. First of all note that J_ε is into X. Due to do this we have the following lemma.

Lemma 7. *Assume $\phi \in X$. Then*

$$\sup_{A_s} |\phi| \le C_1 \frac{\|\phi\|_X}{s^\sigma}, \qquad \sup_{A_s} |\nabla\phi| \le C_1 \frac{\|\phi\|_X}{s^{\sigma+1}}. \tag{21}$$

Proof. By a standard scaling argument and the Sobolev embedding theorem after noting the fact that $N < t < \infty$. □

With respect to Lemma 7, one can conclude if $\phi \in B_R \in X$ then

$$\sup_{A_s} |\phi| \le \frac{C_1 R}{s^\sigma}, \qquad \sup_{A_s} |\nabla\phi| \le \frac{C_1 R}{s^{\sigma+1}}.$$

By the above notes we can prove the following lemmas.

Lemma 8. *Let $\phi \in B_R \subset X$. Then there exists C_{11} such that $\|\frac{F_1(\phi)}{|\nabla w|^2}\|_Y^t \le C_{11}\|\phi\|_X^{2t}$.*

Proof. $\|\frac{F_1(\phi)}{|\nabla w|^2}\|_y^t = \|\frac{\Delta w|\nabla\phi|^2}{|\nabla w|^2}\|_Y^t$ and

$$\begin{aligned}
\|\tfrac{\Delta w|\nabla\phi|^2}{|\nabla w|^2}\|_Y^t &= \sup\nolimits_{0<s\le\frac{1}{2}} s^{(2+\sigma)t-N} \textstyle\int_{A_s} |\tfrac{\Delta w|\nabla\phi|^2}{|\nabla w|^2}(x)|^t dx \\
&= \sup\nolimits_{0<s\le\frac{1}{2}} s^{(2+\sigma)t-N} \textstyle\int_{A_s} |\sigma(\sigma+2-N)|x|^\sigma|\nabla\phi|^2|^t dx \\
&\le C_{11}\|\phi\|_X^{2t}.
\end{aligned}$$

□

Lemma 9. *Let $\phi \in B_R \subset X$. Then there exists C_{22} such that $\|\frac{F_2(\phi)}{|\nabla w|^2}\|_Y^t \leq C_{22}\|\phi\|_X^{2t}$.*

Proof. Since $\|F_2(\phi)\|_y^t = \|\frac{2\nabla w \nabla\phi\Delta\phi}{|\nabla w|^2}\|_Y^t$ and

$$\begin{aligned}
\|\frac{2\nabla w\nabla\phi\Delta\phi}{|\nabla w|^2}\|_Y^t &= \sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{2\nabla w\nabla\phi\Delta\phi}{|\nabla w|^2}(x)|^t dx \\
&= \sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{2}{\sigma}|x|^{\sigma+1}\nabla\phi\Delta\phi(x)|^t dx \\
&\leq C_{22}\|\phi\|_X^{2t}.
\end{aligned}$$

□

Lemma 10. *Let $\phi \in B_R \subset X$. Then there exists C_{33} such that $\|\frac{F_3(\phi)}{|\nabla w|^2}\|_Y^t \leq C_{33}\|\phi\|_X^{3t}$.*

Proof. Since $\|\frac{F_3(\phi)}{|\nabla w|^2}\|_y^t = \|\frac{(\Delta\phi)|\nabla\phi|^2}{|\nabla w|^2}\|_Y^t$ and

$$\begin{aligned}
\|\frac{(\Delta\phi)|\nabla\phi|^2}{|\nabla w|^2}\|_Y^t &= \sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{(\Delta\phi)|\nabla\phi|^2}{|\nabla w|^2}(x)|^t dx \\
&= \sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{|x|^{2\sigma+2}}{\sigma^2}(\Delta\phi|\nabla\phi|^2)(x)|^t dx \\
&\leq C_{33}\|\phi\|_X^{3t}.
\end{aligned}$$

□

Also it easy to see that

Lemma 11. *Let $\phi \in B_R \subset X$. Then there exists C_{456} such that*

$$\begin{aligned}
\|\frac{F_4(\phi)}{|\nabla w|^2}\|_Y^t &= \|\frac{\frac{p-2}{2}\nabla w\cdot\nabla(|\nabla\phi|^2)}{|\nabla w|^2}\|_Y^t \leq C_{456}\|\phi\|_X^{2t}, \\
\|\frac{F_5(\phi)}{|\nabla w|^2}\|_Y^t &= \|\frac{(p-2)\nabla\phi\cdot\nabla(\nabla w\cdot\nabla\phi)}{|\nabla w|^2}\|_Y^t \leq C_{456}\|\phi\|_X^{2t}, \\
\|\frac{F_6(\phi)}{|\nabla w|^2}\|_Y^t &= \|\frac{\frac{p-2}{2}\nabla\phi\cdot\nabla(|\nabla\phi|^2)}{|\nabla w|^2}\|_Y^t \leq C_{456}\|\phi\|_X^{3t}.
\end{aligned}$$

Lemma 12. *There exists C_{77} such that $\|\frac{F_7(\phi)}{|\nabla w|^2}\|_Y^t \leq C_{77}(\|\phi\|_X^{2t} + \|\phi\|_X^{(q-p+4)t})$.*

Proof. with respect to part (2) of Lemma 15 (in Appendix) one can show that

$$\begin{aligned}
\|\frac{F_7(\phi)}{|\nabla w|^2}\|_Y^t &= \left\|C\frac{|\nabla w+\nabla\phi|^{q-p+4}-|\nabla w|^{q-p+4}-(q-p+4)|\nabla w|^{q-p+2}\nabla w\nabla\phi}{|\nabla w|^2}\right\|_Y^t \\
&\leq C_p\left\|\frac{|\nabla\phi|^{q-p+4}+|\nabla w|^{q-p+2}|\nabla\phi|^2}{|\nabla w|^2}\right\|_Y^t \\
&\leq C_p\left\|\frac{|\nabla\phi|^{q-p+4}}{|\nabla w|^2}\right\|_Y^t + C_p\left\|\frac{|\nabla w|^{q-p+2}|\nabla\phi|^2}{|\nabla w|^2}\right\|_Y^t.
\end{aligned}$$

For the first term, we have

$$\left\|\frac{|\nabla\phi|^{q-p+4}}{|\nabla w|^2}\right\|_Y^t = sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{|\nabla\phi|^{q-p+4}}{|\nabla w|^2}|^t dx$$

$$= sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{|x|^{2\sigma+2}}{\sigma^2}|\nabla\phi|^{q-p+4}(x)|^t dx$$

$$\leq C_{74}\|\phi\|_X^{(q-p+4)t}.$$

Since $\sigma=\frac{p-q}{q-p+1}$ and $(\sigma+1)(q-p+4)-2\sigma-2=\sigma+2$. Also

$$\left\|\frac{|\nabla w|^{q-p+2}|\nabla\phi|^2}{|\nabla w|^2}\right\|_Y^t = sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{|\nabla w|^{q-p+2}|\nabla\phi|^2}{|\nabla w|^2}|^t dx$$

$$= sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{\sigma^{q-p}}{|x|^{(\sigma+1)(q-p)}}|\nabla\phi|^2|^t dx$$

$$\leq C_7\|\phi\|_X^{2t}\ sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{\sigma^{q-p}}{|x|^{(\sigma+1)(q-p)}}|x|^{-2(\sigma+1)}|^t dx$$

$$= C_7\|\phi\|_X^{2t}\ sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{\sigma^{q-p}}{|x|^{(\sigma+1)(q-p)+2\sigma+2}}|^t dx$$

$$= C_7\|\phi\|_X^{2t}\ sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{\sigma^{q-p}}{|x|^{\sigma(q-p+1)+q-p+\sigma+2}}|^t dx$$

$$\leq 7C_{77}\|\phi\|_X^{2t}.$$

Since $\sigma=\frac{p-q}{q-p+1}$ and $\sigma(q-p+1)+q-p+\sigma+2=\sigma+2$. □

Finally we have

Lemma 13. *Let $\phi\in B_R\subset X$. Then there exists C_8 such that*

$$\|\frac{I_\varepsilon(\phi)}{|\nabla w|^2}\|_Y^t = \|C\frac{|\nabla w+\nabla\phi+\varepsilon A_0(\nabla w+\nabla\phi)|^{q-p+4}-|\nabla w+\nabla\phi|^{q-p+4}}{|\nabla w|^2}\|_Y^t \leq C_8\varepsilon^t(1+\|\phi\|_X^{(q-p+4)t}).$$

Proof.

$$\|\frac{I_\varepsilon(\phi)}{|\nabla w|^2}\|_Y^t = \|C\frac{|\nabla w+\nabla\phi+\varepsilon A_0(\nabla w+\nabla\phi)|^{q-p+4}-|\nabla w+\nabla\phi|^{q-p+4}}{|\nabla w|^2}\|_Y^t$$

$$\leq C_{81}\varepsilon^t(\||\nabla w|^{q-p+2}\|_Y^t+\|\frac{|\nabla\phi|^{q-p+4}}{|\nabla w|^2}\|_Y^t)$$

$$\leq C_8\varepsilon^t(1+\|\phi\|_X^{(q-p+4)t}).$$

□

Lemma 14. *Suppose $\phi\in B_R\subset X$, then*

$$\|\frac{H_\varepsilon(w+\phi)}{|\nabla w|^2}\|_Y^t \leq C_9\varepsilon^t.$$

Proof. Note that

$$\|\frac{H_\varepsilon(w+\phi)}{|\nabla w|^2}\|_Y^t := \quad sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N}\int_{A_s}|\frac{1}{|\nabla w|^2}\{(\Delta(w+\phi))2\varepsilon(A_0\nabla(w+\phi))\cdot\nabla(w+\phi)$$

$$+\varepsilon^2(\Delta(w+\phi))|A_0\nabla(w+\phi)|^2+E_\varepsilon((w+\phi))|\nabla(w+\phi)|^2$$

$$+E_\varepsilon((w+\phi))(2\varepsilon A_0\nabla(w+\phi))\cdot\nabla(w+\phi)+E_\varepsilon(w+\phi)\varepsilon^2|A_0\nabla(w+\phi)|^2$$

$$+g_0(\varepsilon)\sum\left\{(w+\phi)_{x_ix_j}(w+\phi)_{x_i}(w+\phi)_{x_j}+(w+\phi)_{x_i}(w+\phi)_{x_j}(w+\phi)_{x_k}\right\}\}|^t dx.$$

The above equation include the following terms

$$\frac{\Delta w|\nabla w|^2}{|\nabla w|^2},\frac{\Delta\phi|\nabla w|^2}{|\nabla w|^2},\frac{\Delta w|\nabla\phi|^2}{|\nabla w|^2},\frac{\Delta\phi|\nabla\phi|^2}{|\nabla w|^2},\frac{\Delta w\nabla w\cdot\nabla\phi}{|\nabla w|^2},\frac{\Delta\phi\nabla w\cdot\nabla\phi}{|\nabla w|^2},$$

$$\frac{E_\varepsilon(w)|\nabla w|^2}{|\nabla w|^2},\frac{E_\varepsilon(w)|\nabla\phi|^2}{|\nabla w|^2},\frac{E_\varepsilon(w)\nabla w\cdot\nabla\phi}{|\nabla w|^2},\frac{E_\varepsilon(\phi)|\nabla w|^2}{|\nabla w|^2},\frac{E_\varepsilon(\phi)|\nabla\phi|^2}{|\nabla w|^2},\frac{E_\varepsilon(\phi)\nabla w\cdot\nabla\phi}{|\nabla w|^2},$$

$$\frac{w_{x_ix_j}w_{x_i}w_{x_j}}{|\nabla w|^2},\frac{w_{x_ix_j}w_{x_i}\phi_{x_j}}{|\nabla w|^2},\frac{w_{x_ix_j}\phi_{x_i}\phi_{x_j}}{|\nabla w|^2},\frac{\phi_{x_ix_j}\phi_{x_i}\phi_{x_j}}{|\nabla w|^2},\frac{w_{x_i}w_{x_i}w_{x_j}}{|\nabla w|^2},\frac{w_{x_i}w_{x_i}\phi_{x_j}}{|\nabla w|^2},\frac{w_{x_i}\phi_{x_i}\phi_{x_j}}{|\nabla w|^2},\frac{\phi_{x_i}\phi_{x_i}\phi_{x_j}}{|\nabla w|^2}.$$

Similar to the last lemmas, a computation shows that $\|$each term$\|_Y^t$ is bounded. Definition of $H_\varepsilon(w+\phi)$ shows one can factor ε^t out. Thus ε^t times the last estimated bound of each term will give us the desired result. □

Theorem 4. *Assume $\phi\in B_R\subset X$. Then the following estimates holds:*

(I) $\|\frac{F_1(\phi)}{|\nabla w|^2}\|_Y^t\leq C_{11}\|\phi\|_X^{2t}$.

(II) $\|\frac{F_2(\phi)}{|\nabla w|^2}\|_Y^t\leq C_{22}\|\phi\|_X^{2t}$.

(III) $\|\frac{F_3(\phi)}{|\nabla w|^2}\|_Y^t\leq C_{33}\|\phi\|_X^{3t}$.

(IV) $\|\frac{F_4(\phi)}{|\nabla w|^2}\|_Y^t\leq C_{456}\|\phi\|_X^{2t}$.

(V) $\|\frac{F_5(\phi)}{|\nabla w|^2}|_Y^t\leq C_{456}\|\phi\|_X^{2t}$.

(VI) $\|\frac{F_6(\phi)}{|\nabla w|^2}\|_Y^t\leq C_{456}\|\phi\|_X^{3t}$.

(VII) $\|\frac{F_7(\phi)}{|\nabla w|^2}\|_Y^t\leq C_{77}(\|\phi\|_X^{2t}+\|\phi\|_X^{(q-p+4)t})$

(VII) $\|\frac{I_\varepsilon(\phi)}{|\nabla w|^2}\|_Y^t\leq C_8\varepsilon^t(1+\|\phi\|_X^{(q-p+4)t})$.

(VIII) $\|\frac{H_\varepsilon(w+\phi)}{|\nabla w|^2}\|_Y^t\leq C_9\varepsilon^t$ *for all $\phi\in B_R$ with $R\leq 1$.*

Proof. The proof is straightforward of the Lemmas 8-14. □

Combining the above results we see that for $0 < R < 1$ chosen sufficiently small and then $\varepsilon > 0$ chosen sufficiently small we have $J_\varepsilon(B_R) \subset B_R$.

Contraction: We want to show that for small enough $\varepsilon > 0$ that J_ε is a contraction on $B_R \subset X$ for suitably (small) R. Let $J_\varepsilon(\phi) = \psi$ and $J_\varepsilon(\phi_0) = \psi_0$ with $\phi, \phi_0 \in B_r$. Note that

$$\tilde{L}(\psi) - \tilde{L}(\psi_0) = \quad \sum_{k=1}^{7} \frac{F_k(\phi) - F_k(\phi_0)}{|\nabla w|^2} + \frac{I_\varepsilon(\phi) - I_\varepsilon(\phi_0)}{|\nabla w|^2} + \frac{H_\varepsilon(w+\phi) - H_\varepsilon(w+\phi_0)}{|\nabla w|^2}. \tag{22}$$

Theorem 5. *$J_\varepsilon : B_R \to B_R$ is a contraction, where ε and R are small enough.*

Proof. We have to show that for sufficiently small ε and R, $J_\varepsilon : B_R \to B_R$ is a contraction. In other words we need to show there exists a $k_{R,\varepsilon} < 1$ such that

$$\|J_t(\phi) - J_t(\phi_0)\|_Y \le k_{R,\varepsilon} \|\phi - \phi_0\|_X.$$

We need to prove there exist $k_{R,\varepsilon}$ such that

$$\begin{aligned} &\left\| \frac{F_k(\phi) - F_k(\phi_0)}{|\nabla w|^2} \right\|_Y \le k_{R,\varepsilon} \|\phi - \phi_0\|_X \qquad && \text{for } k = 1, 2, \cdots, 7 \\ &\left\| \frac{I_\varepsilon(\phi) - I_\varepsilon(\phi_0)}{|\nabla w|^2} \right\|_Y \le k_{R,\varepsilon} \|\phi - \phi_0\|_X && \text{and} \\ &\left\| \frac{H_\varepsilon(w+\phi) - H_\varepsilon(w+\phi_0)}{|\nabla w|^2} \right\|_Y \le k_{R,\varepsilon} \|\phi - \phi_0\|_X . \end{aligned} \tag{23}$$

Each of the above inequalities are studied in the following Steps 1-9:
Step 1. $k = 1$. We have

$$\begin{aligned} \| \tfrac{F_1(\phi) - F_1(\phi_0)}{|\nabla w|^2} \|_Y^t = & \ \| \tfrac{\Delta w}{|\nabla w|^2} \left(|\nabla \phi|^2 - |\nabla \phi_0|^2 \right) \|_Y^t \\ = & \ sup_{0 < s \le \frac{1}{2}} s^{(2+\sigma)t - N} \int_{A_s} | \tfrac{\Delta w}{|\nabla w|^2} \left(|\nabla \phi|^2 - |\nabla \phi_0|^2 \right) |^t dx \\ = & \ sup_{0 < s \le \frac{1}{2}} s^{(2+\sigma)t - N} \int_{A_s} | \tfrac{\sigma + 2 - N}{\sigma} |x|^\sigma \left(|\nabla \phi|^2 - |\nabla \phi_0|^2 \right) |^t dx \\ = & \ sup_{0 < s \le \frac{1}{2}} s^{(2+\sigma)t - N} \int_{A_s} | \tfrac{\sigma + 2 - N}{\sigma} |x|^\sigma \left(|\nabla \phi| + |\nabla \phi_0| \right) \left(|\nabla \phi| - |\nabla \phi_0| \right) |^t dx \\ \le & \ (C_1 \tfrac{\sigma + 2 - N}{\sigma} R)^t \|\phi - \phi_0\|_X^t. \end{aligned}$$

Step 2. $k = 2$. We need to show

$$\left\| \frac{F_2(\phi) - F_2(\phi_0)}{|\nabla w|^2} \right\|_Y \le k_{R,\varepsilon} \|\phi - \phi_0\|_X .$$

By the definition we have

$$
\begin{aligned}
\left\| \frac{F_2(\phi)-F_2(\phi_0)}{|\nabla w|^2} \right\|_Y^t &= \left\| \frac{2\nabla w(\nabla\phi\Delta\phi - \nabla\phi_0\Delta\phi_0)}{|\nabla w|^2} \right\|_Y^t \\
&= \left\| \left(\frac{2\nabla w}{|\nabla w|^2} \left(\nabla\phi - \nabla\phi_0\right)\Delta\phi - \left(\Delta\phi_0 - \Delta\phi\right)\nabla\phi_0 \right) \right\|_Y^t \\
&\leq k_{1t} \left\{ \left\| \frac{2\nabla w}{|\nabla w|^2} \left(\nabla\phi - \nabla\phi_0\right)\Delta\phi \right\|_Y^t + \left\| \frac{2\nabla w}{|\nabla w|^2} \left(\Delta\phi_0 - \Delta\phi\right)\nabla\phi_0 \right\|_Y^t \right\} \\
&=: k_{12t} \left(K_{21}(\phi,\phi_0) + K_{22}(\phi,\phi_0) \right).
\end{aligned}
$$

A computation shows for each each term we have as:

$$
\begin{aligned}
K_{21}(\phi,\phi_0) = &\ \sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N} \int_{A_s} \left| \frac{2\nabla w}{|\nabla w|^2} \left(\nabla\phi - \nabla\phi_0\right)\Delta\phi \right|^t dx \\
\leq &\ k_{2t} \left(\sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N} \int_{A_s} |\Delta\phi|^t \right) \left(\|\phi - \phi_0\|_X^t \right) \\
\leq &\ k_{2t} R^t \|\phi - \phi_0\|_X^t.
\end{aligned}
$$

where we have applied Sobolev embedding $s^{(\sigma+1)} \sup_{A_s} |\nabla\phi| \leq C_1 \|\phi\|_X$. For $K_{22}(\phi,\phi_0)$ we have

$$
\begin{aligned}
K_{22}(\phi,\phi_0) = &\ \sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N} \int_{A_s} \left| \frac{2\nabla w}{-w_r^2} \left(\Delta\phi_0 - \Delta\phi\right)\nabla\phi_0 \right|^t dx \\
\leq &\ K_{2t} R^t \sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N} \int_{A_s} \left| \left(\Delta\phi_0 - \Delta\phi\right) \right|^t dx \\
\leq &\ k_{2t} R^t \left\|\phi - \phi_0\right\|_X^t.
\end{aligned}
$$

Consequently

$$
\left\| \frac{F_2(\phi)-F_2(\phi_0)}{|\nabla w|^2} \right\|_Y^t \leq \ k_{2t} R^t \left\|\phi - \phi_0\right\|_X^t.
$$

Step 3. $k = 3$. We have

$$
\begin{aligned}
\left\| \frac{F_3(\phi)-F_3(\phi_0)}{|\nabla w|^2} \right\|_Y^t &= \left\| \frac{(\Delta\phi)|\nabla\phi|^2 - (\Delta\phi_0)|\nabla\phi_0|^2}{|\nabla w|^2} \right\|_Y^t \\
&= \sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N} \int_{A_s} \left| \frac{(\Delta\phi)|\nabla\phi|^2 - (\Delta\phi_0)|\nabla\phi_0|^2}{|\nabla w|^2} \right|^t dx \\
&= \sup_{0<s\leq\frac{1}{2}} s^{(2+\sigma)t-N} \int_{A_s} \left| \frac{|x|^{2\sigma+2}}{\sigma^2} \left((\Delta\phi - \Delta\phi_0)|\nabla\phi|^2 + (\nabla\phi - \nabla\phi_0)(\nabla\phi + \nabla\phi_0)\Delta\phi_0 \right) \right|^t dx \\
&\leq k_{3t} R^{2t} \|\phi - \phi_0\|_X^t.
\end{aligned}
$$

Step 4. $k=4$. We have

$$\left\|\frac{F_k(\phi)-F_k(\phi_0)}{|\nabla w|^2}\right\|_Y^t = \left\|\frac{\frac{p-2}{2}\nabla w}{|\nabla w|^2}\cdot\left(\nabla(|\nabla\phi|^2)-\nabla(|\nabla\phi_0|^2)\right)\right\|_Y^t$$

$$= \left\|\frac{\frac{p-2}{2}\nabla w}{|\nabla w|^2}\cdot\nabla(|\nabla\phi|^2-|\nabla\phi_0|^2)\right\|_Y^t$$

$$\leq \left\|(p-2)\frac{|\nabla w||\nabla\phi||D^2(\phi-\phi_0)|+|\nabla w||D^2\phi_0||\nabla\phi-\nabla\phi_0|}{|\nabla w|^2}\right\|_Y^t$$

$$\leq k_{4t}R^t\|\phi-\phi_0\|_X^t,$$

where we applied Lemma 16 in Appendix.

Step 5. $k=5$. We have

$$\left\|\frac{F_5(\phi)-F_5(\phi_0)}{|\nabla w|^2}\right\|_Y = \|(p-2)\frac{\nabla\phi\cdot\nabla(\nabla w\cdot\nabla\phi)-\nabla\phi_0\cdot\nabla(\nabla w\cdot\nabla\phi_0)}{|\nabla w|^2}\|_Y^t$$

$$= \|(p-2)\frac{|D^2w||\nabla\phi||\nabla\phi-\nabla\phi_0|+|\nabla w||D^2\phi||\nabla\phi-\nabla\phi_0|+|\nabla\phi_0||D^2w||\nabla\phi-\nabla\phi_0|+|\nabla\phi_0||\nabla w||D^2(\phi-\phi_0)|}{|\nabla w|^2}\|_Y^t$$

$$\leq k_{6t}R^t\|\phi-\phi_0\|_X^t.$$

Step 6. $k=6$. We have

$$\left\|\frac{F_6(\phi)-F_6(\phi_0)}{|\nabla w|^2}\right\|_Y = \|\frac{p-2}{2}\frac{\nabla\phi\cdot\nabla(|\nabla\phi|^2)-\nabla\phi_0\cdot\nabla(|\nabla\phi_0|^2)}{|\nabla w|^2}\|_Y^t$$

$$\leq k_{60}\left\|(p-2)\frac{|\nabla\phi_0|^2|D^2(\phi-\phi_0)|+|D^2\phi_0||\nabla\phi+\nabla\phi_0||\nabla\phi-\nabla\phi_0|}{|\nabla w|^2}\right\|_Y^t$$

$$\leq k_{6t}R^{2t}\|\phi-\phi_0\|_X^t.$$

Step 7. $k=7$. I mean we need to prove

$$\left\|\frac{F_7(\phi)-F_7(\phi_0)}{|\nabla w|^2}\right\|_Y \leq k_{R,\varepsilon}\left\|\phi-\phi_0\right\|_X,$$

by the definition we have

$$\left\|\frac{F_7(\phi)-F_7(\phi_0)}{|\nabla w|^2}\right\|_Y^t = \left\|\frac{|\nabla w+\nabla\phi|^{q-p+4}-|\nabla w|^{q-p+4}-(q-p+4)|\nabla w|^{q-p+2}\nabla w\nabla\phi}{|\nabla w|^2}\right.$$

$$\left.-\frac{|\nabla w+\nabla\phi_0|^{q-p+4}-|\nabla w|^{q-p+4}-(q-p+4)|\nabla w|^{q-p+2}\nabla w\nabla\phi_0}{|\nabla w|^2}\right\|_Y^t$$

$$= \left\|\frac{|\nabla w+\nabla\phi|^{q-p+4}-|\nabla w+\nabla\phi_0|^{q-p+4}-(q-p+4)|\nabla w|^{q-p+2}\nabla w(\nabla\phi-\nabla\phi_0)}{|\nabla w|^2}\right\|_Y^t$$

$$\leq C\left\|\frac{1}{|\nabla w|^2}\left(|\nabla w|^{q-p+2}\left(|\nabla\phi|+|\nabla\phi_0|\right)+|\nabla\phi|^{q-p+3}+|\nabla\phi_0|^{q-p+3}\right)|\nabla\phi_0-\nabla\phi|\right\|_Y^t,$$

where we applied part (4) of Lemma 15 (in Appendix). Thus with respect to the definition of $\|\cdot\|_Y$ we have

$$
\begin{aligned}
&\left\|\tfrac{1}{|\nabla w|^2}\left(|\nabla w|^{q-p+2}\left(|\nabla\phi|+|\nabla\phi_0|\right)+|\nabla\phi|^{q-p+3}+|\nabla\phi_0|^{q-p+3}\right)|\nabla\phi_0-\nabla\phi|\right\|_Y^t\\
\leq\ & k_{71}\left(\left\|\tfrac{1}{|\nabla w|^2}|\nabla w|^{q-p+2}\left(|\nabla\phi|+|\nabla\phi_0|\right)|\nabla\phi_0-\nabla\phi|\right\|_Y^t\right.\\
&\qquad\left.+\left\|\tfrac{1}{|\nabla w|^2}\left(|\nabla\phi|^{q-p+3}+|\nabla\phi_0|^{q-p+3}\right)|\nabla\phi_0-\nabla\phi|\right\|_Y^t\right)\\
:=\ & k_{71}\left(K_{11}(\phi,\phi_0)+K_{12}(\phi,\phi_0)\right).
\end{aligned}
$$

A computation shows

$$
\begin{aligned}
K_{11}(\phi,\phi_0) =\ & \left\|\tfrac{1}{|\nabla w|^2}|\nabla w|^{q-p+2}\left(|\nabla\phi|+|\nabla\phi_0|\right)|\nabla\phi_0-\nabla\phi|\right\|_Y^t\\
\leq\ & \left(2(C\sigma)^{q-p}C_1R\right)^t\|\phi-\phi_0\|_X^t
\end{aligned}
$$

and

$$
\begin{aligned}
K_{12}(\phi,\phi_0) =\ & \left\|\tfrac{1}{|\nabla w|^2}\left(|\nabla\phi|^{q-p+3}+|\nabla\phi_0|^{q-p+3}\right)|\nabla\phi_0-\nabla\phi|\right\|_Y^t\\
\leq\ & \left(\tfrac{2(C_1R)^{(q-p+3)}}{C^2\sigma^2}\right)^t\|\phi-\phi_0\|_X^t.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left\|\tfrac{F_7(\phi)-F_7(\phi_0)}{|\nabla w|^2}\right\|_Y^t \leq\ & k_{71}\left(K_{11}(\phi,\phi_0)+K_{12}(\phi,\phi_0)\right)\\
\leq\ & k_{71}\left(\left(2(C\sigma)^{q-p}C_1R\right)^t+\left(\tfrac{2(C_1R)^{(q-p+3)}}{C^2\sigma^2}\right)^t\right)\|\phi-\phi_0\|_X^t\\
\leq\ & k_{R,\varepsilon}\|\phi-\phi_0\|_X.
\end{aligned}
$$

Step 8. We need to show

$$
\left\|\frac{I_\varepsilon(\phi)-I_\varepsilon(\phi_0)}{|\nabla w|^2}\right\|_Y \leq k_{R,\varepsilon}\|\phi-\phi_0\|_X.
$$

With respect to part (4) of Lemma 15 (in Appendix) one can write

$$\begin{aligned}
&\|\tfrac{I_\varepsilon(\phi)-I_\varepsilon(\phi_0)}{|\nabla w|^2}\|_Y^t \\
= \;& C^t\|\tfrac{|\nabla w+\nabla\phi+\varepsilon A_0(\nabla w+\nabla\phi)|^{q-p+4}-|\nabla w+\nabla\phi|^{q-p+4}}{|\nabla w|^2}-\tfrac{|\nabla w+\nabla\phi_0+\varepsilon A_0(\nabla w+\nabla\phi_0)|^{q-p+4}-|\nabla w+\nabla\phi_0|^{q-p+4})}{|\nabla w|^2}\|_Y^t \\
\leq \;& k_{81}\Big(\|\tfrac{|\nabla w+\nabla\phi+\varepsilon A_0(\nabla w+\nabla\phi)|^{q-p+4}-|\nabla w+\nabla\phi_0+\varepsilon A_0(\nabla w+\nabla\phi_0)|^{q-p+4}-(1+\varepsilon)(q-p+4)|\nabla w|^{q-p+2}\nabla w(\nabla\phi-\nabla\phi_0)}{|\nabla w|^2}\|_Y^t \\
&+\|\tfrac{|\nabla w+\nabla\phi|^{q-p+4}-|\nabla w+\nabla\phi_0|^{q-p+4}-(q-p+4)|\nabla w|^{q-p+2}\nabla w(\nabla\phi-\nabla\phi_0)}{|\nabla w|^2}\|_Y^t \\
&+\|\tfrac{\varepsilon(q-p+4)|\nabla w|^{q-p+2}\nabla w(\nabla\phi-\nabla\phi_0)}{|\nabla w|^2}\|_Y^t\Big) \\
\leq \;& k_{82}\Big(\|\Big[\tfrac{|\nabla w|^{q-p+2}(|\nabla\phi+\varepsilon(\nabla w+\nabla\phi)|+|\nabla\phi_0+\varepsilon(\nabla w+\nabla\phi_0)|)+|\nabla\phi+\varepsilon(\nabla w+\nabla\phi)|^{q-p+3}}{|\nabla w|^2} \\
&\qquad+\tfrac{|\nabla\phi_0+\varepsilon(\nabla w+\nabla\phi_0)|^{q-p+3}}{|\nabla w|^2}\Big](1+\varepsilon)\,|\nabla\phi-\nabla\phi_0|\|_Y^t \\
&+\|\tfrac{\big[|\nabla w|^{q-p+2}(|\nabla\phi|+|\nabla\phi_0|)+|\nabla\phi|^{q-p+3}+|\nabla\phi_0|^{q-p+3}\big]|\nabla\phi-\nabla\phi_0|}{|\nabla w|^2}\|_Y^t \\
&+\|\tfrac{\varepsilon(q-p+4)|\nabla w|^{q-p+2}|\nabla w||\nabla\phi-\nabla\phi_0|}{|\nabla w|^2}\|_Y^t\Big) \\
\leq \;& k_{83}\big[(1+\varepsilon)^t\left\{R^t+\varepsilon^t(1+R^t)+R^{(q-p+3)t}+\varepsilon^{(q-p+3)t}(1+R^{(q-p+3)t})\right\} \\
&\qquad+(R^t+R^{(q-p+3)t})+\varepsilon^t(q-p+4)^t\big]\,\|\phi-\phi_0\|_X^t.
\end{aligned}$$

Or it means that

$$\|\frac{I_\varepsilon(\phi)-I_\varepsilon(\phi_0)}{|\nabla w|^2}\|_Y \leq k_{R,\varepsilon}\|\phi-\phi_0\|_X.$$

Step 9. We need to show that

$$\left\|\frac{H_\varepsilon(w+\phi)-H_\varepsilon(w+\phi_0)}{|\nabla w|^2}\right\|_Y \leq k_{R,\varepsilon}\left\|\phi-\phi_0\right\|_X.$$

We know that

$$\left\| \frac{H_\varepsilon(w+\phi)-H_\varepsilon(w+\phi_0)}{|\nabla w|^2} \right\|_Y^t$$

$$= \left\| \frac{1}{|\nabla w|^2} \Big[\right.$$

$$2\varepsilon \left(\Delta(w+\phi)(A_0\nabla(w+\phi))\cdot\nabla(w+\phi) - \Delta(w+\phi_0)(A_0\nabla(w+\phi_0))\cdot\nabla(w+\phi_0)\right)$$

$$+\varepsilon^2 \left(\Delta(w+\phi)|A_0\nabla(w+\phi)|^2 - \Delta(w+\phi_0)|A_0\nabla(w+\phi_0)|^2\right)$$

$$+E_\varepsilon((w+\phi))|\nabla(w+\phi)|^2 - E_\varepsilon((w+\phi_0))|\nabla(w+\phi_0)|^2$$

$$+2\varepsilon \left(E_\varepsilon(w+\phi)(A_0\nabla(w+\phi))\cdot\nabla(w+\phi) - E_\varepsilon(w+\phi_0)(A_0\nabla(w+\phi_0))\cdot\nabla(w+\phi_0)\right)$$

$$+\varepsilon^2 \left(E_\varepsilon(w+\phi)|A_0\nabla(w+\phi)|^2 - E_\varepsilon(w+\phi_0)|A_0\nabla(w+\phi_0)|^2\right)$$

$$+g_0(\varepsilon)\left\{\sum\left\{(w+\phi)_{x_ix_j}(w+\phi)_{x_i}(w+\phi)_{x_j} + (w+\phi)_{x_i}(w+\phi)_{x_j}(w+\phi)_{x_k}\right\}\right.$$

$$\left. - \sum\left\{(w+\phi_0)_{x_ix_j}(w+\phi_0)_{x_i}(w+\phi_0)_{x_j} + (w+\phi_0)_{x_i}(w+\phi_0)_{x_j}(w+\phi_0)_{x_k}\right\}\right\} \Big] \|_Y^t.$$

Similar to the other cases (which is done above), a computation shows each of the above differences is bounded by $k_9\varepsilon^t\|\phi-\phi_0\|_X^t$.

□

Note to see the solution v is positive we note that $v(x) = w(x)+\phi(x)$ and then note by taking $R>0$ small enough and using the pointwise estimate on ϕ and its gradient that we have $v>0$ in B_1.

3.2 The Hölder continues case

To apply the fixed point theorem in this section we argue exactly as in the previous section. In other words, by the same argument one can show that for $0<R<1$ chosen sufficiently small and then $\varepsilon>0$ chosen sufficiently small we have $J_\varepsilon(B_R)\subset B_R$, and for small enough $\varepsilon>0$, J_ε is a contraction on $B_R\subset X$ for suitably (small) R.

4 Appendix

Here we recall the following lemma which is necessary to follow the problem.

Lemma 15. *Suppose $p>1$. There exists a constant C such that the following hold:*

(1) For all numbers $w>0$ and $\phi,\tilde\phi\in\mathbb{R}$,

$$\left||w+\phi|^p - pw^{p-1}\phi - w^p\right| \le C\left(w^{p-2}\phi^2+|\phi|^p\right),$$

and

$$\left||w+\tilde\phi|^p - |w+\phi|^p - pw^{p-1}(\tilde\phi-\phi)\right| \le C\left(w^{p-2}\left(|\phi|+|\tilde\phi|\right)+|\phi|^{p-1}+|\tilde\phi|^{p-1}\right)\left|\tilde\phi-\phi\right|.$$

(2) For all $x, y, z \in \mathbb{R}^N$,

$$||x+y|^p - |x+z|^p| \leq C\left(|x|^{p-1} + |y|^{p-1} + |z|^{p-1}\right)|y-z|.$$

(3) For $p > 1$, there exists C_p such that for and $x, y \in \mathbb{R}^N$ and ($x \neq 0$) and $|y|$ small enough

$$||x+y|^p - |x|^p - p|x|^{p-2}x \cdot y| \leq C_p\left(|y|^p + |x|^{p-2}|y|^2\right).$$

(4) For $x, y, z \in \mathbb{R}^N$,

$$\left||x+y|^p - |x+z|^p - p|x|^{p-2}x \cdot (y-z)\right| \leq C\left(|x|^{p-2}\left(|y|+|z|\right) + |y|^{p-1} + |z|^{p-1}\right)|y-z|.$$

Recall that $F_4(\phi) = \frac{p-2}{2}\nabla w \cdot \nabla(|\nabla\phi|^2)$ and $F_5(\phi) = (p-2)\nabla\phi \cdot \nabla(\nabla w \cdot \nabla\phi)$, then we have the following lemma.

Lemma 16. *Let $\phi_0, \phi_1 \in X$. Then there exist C_1 and C_2 such that*

$$\begin{aligned} |F_4(\phi_1) - F_4(\phi_0)| &\leq C_1\left(|\nabla w||\nabla\phi_1||D^2(\phi_1-\phi_0)| + |\nabla w||D^2\phi_0||\nabla\phi_1 - \nabla\phi_0|\right), \\ |F_5(\phi_1) - F_5(\phi_0)| &\leq C_2\left(|D^2 w||\nabla\phi_1||\nabla\phi_1 - \nabla\phi_0| + |\nabla w||D^2\phi_1||\nabla\phi_1 - \nabla\phi_0|\right. \\ &\qquad \left. + |\nabla\phi_0||D^2 w||\nabla\phi_1 - \nabla\phi_0| + |\nabla\phi_0||\nabla w||D^2(\phi_1 - \phi_0)|\right). \end{aligned} \tag{24}$$

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