SUPERCRITICAL PROBLEMS VIA A FIXED POINT ARGUMENT ON THE CONE OF MONOTONIC FUNCTIONS

C. COWAN

ABSTRACT. Here we develop a general procedure to obtain positive classical solutions of various supercritical elliptic problems on domains of double revolution with certain symmetry and monotonicity properties. The main thrust of the work is that with the availability of suitable lower dimensional Liouville theorems one can use a unified approach in proving existence of solutions.

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1. INTRODUCTION

In this paper we are interested in positive classical solutions of various supercritical elliptic problems on domains which satisfy certain symmetry and monotonicity conditions. In general the domains will be annular domains with certain symmetry and monotonicity. The first problem is a Hénon equation and here the problem is not posed on a annular domain. The main thrust of this work is that the same general procedure gives existence results for a large class of problems. We now list some problems that we can handle with this method provided one has the available Liouville theorems.

- (Hénon equation)
$$\begin{cases} -\Delta u = |x|^\alpha u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$
where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ and where $p > 1$.
- (Gradient systems)
$$\begin{cases} -\Delta u = u^{p-1} v^q & \text{in } \Omega, \\ -\Delta v = u^p v^{q-1} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$
where Ω is a bounded domain in $\mathbb{R}^N$ and where $p, q > 1$.
- (Hamiltonian systems)
$$\begin{cases} -\Delta u = u^p v^{q-1} & \text{in } \Omega, \\ -\Delta v = u^{p-1} v^q & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$
where Ω is a bounded domain in $\mathbb{R}^N$ and where $p, q > 1$.

- (General elliptic system)
$$\begin{cases} -\Delta u = u^{p_1} v^{q_1} & \text{in } \Omega, \\ -\Delta v = u^{p_2} v^{q_2} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$
where Ω is a bounded domain in $\mathbb{R}^N$ and where $p_i, q_i \geq 1$.
- (A Lane-Emden system)
$$\begin{cases} -\Delta u = v^p & \text{in } \Omega, \\ -\Delta v = u^q & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$
where Ω is a bounded domain in $\mathbb{R}^N$ and $p, q > 1$.

Our main interest is not in obtaining optimal results but is rather to give a unified approach to various elliptic problems set on a certain type of domain. Note some of the examples we give have a variational structure but this is not needed for our approach. The restrictions on the various parameters exponents p, q, p_i, q_i will come from the availability of suitable Liouville theorems on lower dimensional objects. Here we give an answer for the case of the Hénon equation and also for the Lane-Emden system. For the other problems the existence is left up to finding a suitable Liouville theorem.

1.1. **Background.** Here we give some background on the the scalar problem
$$\begin{cases} -\Delta u = u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$
We assume Ω a bounded smooth domain in $\mathbb{R}^N$. When $N = 2$ there is a positive smooth solution of (6) for any $p > 2$. For $N \geq 3$ the critical exponent $2^* := \frac{2N}{N-2}$ plays a crucial role and for $2 < p < 2^*$ a variational approach shows the existence of a smooth positive solution of (6). For general domains in the critical/supercritical case, $p \geq 2^*$, the existence versus nonexistence of positive solutions of (6) presents a great degree of difficulties; see [3, 23, 34, 33, 32, 31, 35, 54, 55, 59, 60, 62, 63]. Many of these results are very technical and some require perturbation arguments.

In recent years there has been some interest in restricting the function spaces to ones of monotonic functions when trying to solve elliptic pde's with the hope of this allowing for an increase in compactness. The was first done for the Neumann problems on a ball. Consider
$$\begin{cases} -\Delta u + u = a(r) u^{p-1} & \text{in } B_1, \\ \partial_\nu u = 0 & \text{on } \partial B_1, \end{cases} \tag{7}$$
where B_1 is the unit ball centered at the origin in $\mathbb{R}^N$. The interest here is in obtaining nontrivial solutions for values of $p > \frac{2N}{N-2}$. In [5] they considered the variant of (7) given by $-\Delta u + u = |x|^\alpha u^{p-1}$ in B_1 with $\frac{\partial u}{\partial \nu} = 0$ on ∂B_1 (for Dirichelt versions of the Hénon equation see, for instance, [58, 42, 25]). They proved the existence of a positive radial solutions of this equation with arbitrary growth using a shooting argument. The solution turns out to be an increasing function. They also perform numerical computations to see the existence of positive oscillating

solutions. In [65] they considered (7) along with the classical energy associated with the equation given by

$$E(u) := \int_{B_1} \frac{|\nabla u|^2 + u^2}{2}\, dx - \int_{B_1} a(|x|)F(u)\, dx,$$

where $F'(u) = f(u)$ (they considered a more general nonlinearity). Their goal was to find critical points of E over $H^1_{rad}(B_1) := \{u \in H^1(B_1) : u \text{ is radial}\}$. Of course since f is supercritical the standard approach of finding critical points will present difficulties and hence their idea was to find critical points of E over the cone $\{u \in H^1_{rad}(B_1) : 0 \le u,\ u \text{ increasing}\}$. Note that with u nonnegative and increasing this should give improved compactness results since the only place for the functions to be large is near the boundary (but if one replaced with increasing with decreasing then there is no increased compactness expected). Finding the existence of a minimizer is quite standard but now the issue is the critical points don't necessarily correspond to critical points over $H^1_{rad}(B_1)$ and hence one can't conclude the critical points solve the equation. The majority of their work was to show that in fact the critical points of E on the cone are really critical points over the full space. We mention that this work generated a lot of interest in this equation and many authors investigated these idea's of using monotonicity to overcome a lack of compactness. For further results regarding these Neumann problems on radial domains (some using these monotonicity ideas and some using other new methods) see [45, 10, 64, 8, 11, 6, 7, 9, 52, 53].

In [28] we considered (7) using a new variational principle. We obtained positive solutions assuming the same assumptions as the earlier works. In the case of $a = 1$ our approach allowed one to show the solution was nonntrivial (ie. $u = 1$) in a much easier way than the other methods. In [26] we examined the Neumann problem given by

$$\begin{cases} -\Delta u + u = a(x)f(u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where Ω is a bounded domain in $\mathbb{R}^N$ which was a *domain of m revolution* with certain symmetry and where a also satisfied some symmetry assumptions. For the sake of the background one can take $f(u) = u^{p-1}$ and hence a supercritical result would be if $p > 2* := \frac{2N}{N-2}$. In this case we obtained positive nontrivial monotonic solutions of (8) provided $2 < p < 2^*_m := \frac{2m}{m-2}$. For Neumann problems on general domains see [30, 44, 46, 47, 39, 48, 61, 69].

We now return to the Dirichlet problems. There have been many supercritical works that deal with domains that have certain symmetry, for instance, see [16, 17, 18, 19, 20, 21, 22].

In the case of the annulur domains the authors in [13, 15, 51] examined subcritical or slightly supercritical problems on expanding annuli and obtained nonradial solutions. In [43] they obtain nonradial solutions to supercritical problems on expanding annulur domains. In [4] they consider nonradial expanding annulur domains and they obtain the existence of positive solutions. In [35, 20] they consider domains with a small hole and obtain positive solutions.

We now consider the very recent work [12] where they examined

$$\begin{cases} -\Delta u + u = a(x)|u|^{p-2}u, & \text{in } A, \\ u > 0, & \text{in } A, \\ u = 0, & \text{on } \partial A, \end{cases} \tag{9}$$

where $A := \{x \in \mathbb{R}^N : 0 < R_1 < |x| < R_2 < \infty\}$ and $a(x)$ is positive, even with respect to x_N, axial symmetric with respect to the x_N axis and where θ denotes the angle between the the $\mathbb{R}^{N-1}$ plane and x and where $a = a(r, \theta)$ satisfies $a_\theta \le 0$ in upper half of the A. They work on a convex cone $\mathcal{K}$ which is characterized by monotonicity properties of the functions; the functions are increasing in θ (this idea of monotonicity has been used a lot in Neumann problems but is new for Dirichlet problems). They consider the standard energy functional associated with (9) and they work on $\mathcal{K}$ and they also consider $\mathcal{N}_{\mathcal{K}} := \{u \in \mathcal{K} : I'(u)u = 0\}$ which is a Nehari type set adapted to $\mathcal{K}$. Then then develop a mountain pass type argument that utilizes some technical aspects of an associated flow and then they use some involved arguments from dynamical systems to prove the existence of a solution. They obtain a positive solution for all $p > 2$. In the case of $a(x) = 1$ they obtain results regarding nonradial solutions under certain assumptions on the radii of the annulus or the value of p. We mention that even though we had considered convex sets K, which were monotone in an angle, it was this work [12] that we became aware of the ability for this to help gain compactness.

In [27] we considered

$$\begin{cases} -\Delta u = a(x)u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \ge 3)$ and where $p > 1$ and where a is nonnegative and has a certain monotonicity. We considered annular domains and also annular domains with monotonicity (see Definition 2). Using a new variational principle we showed that one could obtain classical positive solutions for a large range of supercritical values of p. In the case of an annular domain (without monotonicity) one still gets a supercritical range of p but when one adds monotonicity they get a larger range (assuming $n \le m$). The approach was to use a new variational principle developed in [56] (see also [2, 49, 57] for more applications) on a suitable convex cone K (we used an $H^1_{0,G}(\Omega)$ version of the cone given in (18)). The variational approach gave added benefits in that on an annulus it allowed one to easily prove the existence of nonradial solutions. In the case of $m = n$ the monotonicity did not give an increased range of p; but in this case there is a new phenomena. In this case the equation has a certain symmetry that allows us to handle domains that have a new $\frac{\pi}{4}$ symmetry (see Definition 3); this is done in [29]. We finally mention that in [1] we considered Gelfand type problems on these new domains. Here we used a various arguments to show that extremal solution gained regularity. One approach was to use blow up analysis and this blow up analysis is crucial for the work in this current paper.

For our current work we are using a fixed point argument that we first saw in [10] (See Theorem B) and applying it to the K we introduced in [27] and then applying the needed blow up analysis.

1.1.1. *Background on the Hénon equation.* Here we give a brief background of (1). In [58] it was shown for $\alpha > 0$ and $\Omega = B_1$ the unit ball in $\mathbb{R}^N$ centred at the origin in $\mathbb{R}^N$ (with $N \geq 3$) there is a positive radial solution of (1) provided

$$1 < p < p_\alpha(N) := \frac{N+2+2\alpha}{N-2},$$

and note this allows for a range of p which are supercritical. In [66] it was shown that under certain assumptions on the parameters that the ground state solution of (1) is nonradial. This work set off a flurry of activity and many works looked at the Hénon equation.

1.2. **Domains of double revolution.** Unless explicity stated we are always assuming our domains will be domains of double revolution. Consider writing $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ where $m, n \geq 1$ and $m + n = N$. We define the variables s and t by

$$s := \left\{x_1^2 + \cdots + x_m^2\right\}^{\frac{1}{2}}, \qquad t := \left\{x_{m+1}^2 + \cdots + x_N^2\right\}^{\frac{1}{2}}.$$

We say that $\Omega \subset \mathbb{R}^N$ is a *domain of double revolution* if it is invariant under rotations of the first m variables and also under rotations of the last n variables. Equivalently, Ω is of the form $\Omega = \{x \in \mathbb{R}^N : (s,t) \in U\}$ where U is a domain in $\mathbb{R}^2$ symmetric with respect to the two coordinate axes. In fact,

$$U = \left\{(s,t) \in \mathbb{R}^2 : x = (x_1 = s, x_2 = 0, ..., x_m = 0, x_{m+1} = t, ..., x_N = 0) \in \Omega\right\},$$

is the intersection of Ω with the (x_1, x_{m+1}) plane. Note that U is smooth if and only if Ω is smooth. We denote $\widehat{\Omega}$ to be the intersection of U with the first quadrant of $\mathbb{R}^2$, that is,

$$\widehat{\Omega} = \left\{(s,t) \in U : \ s > 0, \ t > 0\right\}. \tag{11}$$

Using polar coordinates we can write $s = r\cos(\theta)$, $t = r\sin(\theta)$ where $r = |x| = |(s,t)|$ and θ is the usual polar angle in the (s,t) plane.

The domains under the consideration will be some domains which have a certain monotonicity in the polar angle θ. All domains will be bounded domains in $\mathbb{R}^N$ with smooth boundary unless otherwise stated. To describe the domains in terms of the above polar coordinates we will write

$$\widetilde{\Omega} := \left\{(\theta, r) : (s,t) \in \widehat{\Omega}\right\}. \tag{12}$$

Definition 1. *We call Ω a* ***monotonic domain of double of revolution*** *provided Ω is a bounded smooth domain of double revolution in $\mathbb{R}^N$ with $N = m+n$ ($n \leq m$) and we can write $\widetilde{\Omega} = \{(\theta, r) : 0 < r < g(\theta), 0 < \theta < \frac{\pi}{2}\}$ for some smooth positive function g on $[0, \frac{\pi}{2}]$ with g decreasing and $g'(0) = g'(\frac{\pi}{2}) = 0$.*

Definition 2. *We refer to a domain of double revolution in $\mathbb{R}^N$ with $N = m+n$ ($n \leq m$) to be an* ***annular domain*** *if*

$$\widetilde{\Omega} = \left\{(\theta, r) : g_1(\theta) < r < g_2(\theta), \theta \in \left(0, \frac{\pi}{2}\right)\right\} \tag{13}$$

$g_i > 0$ is smooth on $[0, \frac{\pi}{2}]$ with $g_i'(0) = g_i'(\frac{\pi}{2}) = 0$ and $g_2(\theta) > g_1(\theta)$ on $[0, \frac{\pi}{2}]$. We will call Ω an ***annular domain with monotonicity*** *if g_1 is increasing and g_2 is decreasing on $(0, \frac{\pi}{2})$.*

Definition 3. *We will call a domain of double revolution in $\mathbb{R}^N$ a* $\frac{\pi}{4}$***-annular domain with monotonicity*** *provided $m = n$, $g_i > 0$ is smooth on $[0, \frac{\pi}{2}]$ with $g_i'(0) = g_i'(\frac{\pi}{2}) = 0$ and $g_2(\theta) > g_1(\theta)$ on $[0, \frac{\pi}{2}]$) and g_1 is increasing and g_2 is* *decreasing on* $(0, \frac{\pi}{4})$ *and both* g_1, g_2 *are even across* $\theta = \frac{\pi}{4}$. *For these new domains we define a suitable subset of* $\widetilde{\Omega}$ *given by*

$$\widetilde{\Omega}_0 = \left\{ (\theta, r) : g_1(\theta) < r < g_2(\theta), 0 < \theta < \frac{\pi}{4} \right\}. \tag{14}$$

1.3. **Main results.**

Theorem 1. *Suppose* Ω *is a monotonic domain of double revolution and suppose*

$$1 < p < \min\left\{ \frac{N+2+2\alpha}{N-2}, \frac{n+3}{n-1} \right\}.$$

Then there is a positive classical solution of (1).

Theorem 2.

(1) *Suppose* Ω *is an annular domain with monotonicity or a* $\frac{\pi}{4}$*-annular domain with monotonicity. Assuming the availability of suitable Liouville theorems for (2), (3), (4) or (5) on* $\mathbb{R}^{n+1}$ *or related half spaces; then there is a nonnegative nonzero classical solution of (2), (3), (4) or (5).*

(2) *(Lane-Emden system, (5)). Suppose* Ω *is an annular domain with monotonicity or a* $\frac{\pi}{4}$*-annular domain with monotonicity. Suppose* $3 = n \le m$ *and*

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{n+1} = \frac{1}{2},$$

then there is a positive classical solution of (5).

Note in part (2) of the above theorem we chose $n = 3$ since this is the case where the optimal Liouville theorem is known. Of course for other dimensions n we have results that allow supercritical values of p, q.

Remark 1.

(1) *One can consider much more general problems than we are considering here at essentially no extra cost. Firstly one can insert appropriate functions of the form* $a(x)$ *on the right hand side of (1), (2), (3), (4) provided* $a \ge 0$ *in* $\widehat{\Omega}$ *and* $a_\theta \le 0$ *in* $\widetilde{\Omega}$. *One can generally add suitable lower order terms; in the blow up analysis they will disappear.*

(2) *As mentioned the main thrust of this work is to not obtain optimal results. If one wants optimal results this will generally require optimal Liouville theorems. We note these results strongly depend on the exact form the equation we are looking at and obtaining optimal Liouville theorems is currently a very active field of research. We give two examples here where we follow through with applying known Liouville theorems; the Hénon equation case and the Lame-Emden system case.*

(3) *In Theorem 2 we include the case of* $\frac{\pi}{4}$*-annular domain with monotonicity but we do not give a proof for this case. The main issue here is that when* $m = n$ *there is an added symmetry in the equation about* $\theta = \frac{\pi}{4}$ *(see (24) for instance that* κ *is odd about* $\theta = \frac{\pi}{4}$ *when* $m = n$*) and this allows one to prove extra results. In the scalar case with a* $\frac{\pi}{4}$*-annular domain with monotonicity we define* K *to be the set of* $0 \le u \in C^1(\overline{\Omega}) \cap H^1_{0,G}(\Omega)$ *which are even about* $\theta = \frac{\pi}{4}$ *in* $\widetilde{\Omega}$ *and such that* $u_\theta \le 0$ *for* $0 < \theta < \frac{\pi}{4}$. *See* [29] *for more details on this type of domain and an approach to prove Lemma 1 for this new* K.

2. Elliptic problems on domains of double revolution

We shall begin by providing some more background on quantities related to domains of double revolution that are essential in this work. Assume Ω is a domain of double revolution and v is a function defined on Ω that just depends on (s,t), then one has

$$\int_\Omega v(x)dx = c(m,n)\int_{\widehat{\Omega}} v(s,t)s^{m-1}t^{n-1}dsdt,$$

where $c(m,n)$ is a positive constant depending on n and m. Set $d\mu(s,t)=s^{m-1}t^{n-1}dsdt$. Note that strictly speaking we are abusing notation here by using the same name; and we will continuously do this in this article. Given a function v defined on Ω we will write $v=v(s,t)$ to indicate that the function has this symmetry. Define

$$H^1_{0,G} := \left\{u \in H^1_0(\Omega) : gu = u \quad \forall g \in G\right\},$$

where $G := O(m)\times O(n)$ where $O(k)$ is the orthogonal group in $\mathbb{R}^k$ and $gu(x) := u(g^{-1}x)$.

To solve equations on domains of double revolution one needs to relate the equation to a new one on $\widehat{\Omega}$ defined in (11). Suppose Ω is a domain of double revolution and f has is function defined on Ω with the same symmetry (ie. $gf(x)=f(g^{-1}x)$ all $g\in G$). Suppose that $u(x)$ solves

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Then $u=u(s,t)$ and u solves

$$-u_{ss}-u_{tt}-\frac{(m-1)u_s}{s}-\frac{(n-1)u_t}{t} = f(s,t) \text{ in } \widehat{\Omega}, \tag{16}$$

with $u=0$ on $(s,t)\in\partial\widehat{\Omega}\backslash(\{s=0\}\cup\{t=0\})$. If u is sufficiently smooth then $u_s=0$ on $\partial\widehat{\Omega}\cap\{s=0\}$ and $u_t=0$ on $\partial\widehat{\Omega}\cap\{t=0\}$ after considering the symmetry properties of u.

2.1. **An outline of the approach.** To illustrate the approach we consider finding a positive solution of

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

where Ω is an annular domain with monotonicity in $\mathbb{R}^N$. Define the set K by

$$K := \left\{0 \le u \in C^1(\overline{\Omega}) \cap H^1_{0,G}(\Omega) : u_\theta \le 0 \text{ a.e. in } \widetilde{\Omega}\right\}. \tag{18}$$

We define the nonlinear mapping T by: given $u\in K$ set $v=T(u)$ where v solves

$$\begin{cases} -\Delta v(x) = u(x)^p & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega. \end{cases} \tag{19}$$

In terms of (s,t) we have v solves

$$\begin{cases} -v_{ss}-v_{tt}-\frac{(m-1)v_s}{s}-\frac{(n-1)v_t}{t} = u(s,t)^p \quad \text{in } \widehat{\Omega}, \end{cases} \tag{20}$$

with $v=0$ on $(s,t)\in\partial\widehat{\Omega}\backslash(\{s=0\}\cup\{t=0\})$ and $v_s=0$ on $\partial\widehat{\Omega}\cap\{s=0\}$ and $v_t=0$ on $\partial\widehat{\Omega}\cap\{t=0\}$.

To find a solution we hope to find some $u \in K\backslash\{0\}$ with $T(u) = u$. We will apply the following result.

Theorem B. [50] *Let K denote a closed convex cone in Banach space X and suppose $T : K \to K$ is completely continuous and set $K(a, b) = \{u \in K : a \le \|u\| \le b\}$ where $0 < a < b < \infty$ and set $K_a = \{u \in K : \|u\| = a\}$ and similarly for K_b. Suppose:*

(i) $\forall \lambda > 1, \forall u \in K_a$ one has $T(u) \neq \lambda u$,

(ii) $\exists w \in K\backslash\{0\}$, $\forall \lambda \ge 0, \forall u \in K_b$ one has $u - T(u) \neq \lambda w$.

Then there is some $u \in K(a, b)$ such that $T(u) = u$.

We mention that we became aware of Theorem B and the applicability of this to supercritical elliptic pde via the work [10] where they used this fixed point theorem to consider radial increasing Neumann problems.

To apply the result we first need to show that $T(K) \subset K$ and we define the norm by $\|u\| = \sup_\Omega |\nabla u(x)|$. Its clear that $0 \le v \in C^1(\overline{\Omega}) \cap H^1_{0,G}(\Omega)$and hence the main thing to show is that $v_\theta \le 0$ a.e. in $\widetilde{\Omega}$. This will follow by a maximum principle type argument but one has to be a bit careful with some singularities of the equation. Crucial to showing this result is the monotonicity assumptions on Ω; the result in general is false.

We now try and verify (i) and (ii). If (i) does not hold then there is some $\lambda > 1$ such that $T(u) = \lambda u$ and hence $-\Delta(\lambda u) = u^p$ in Ω with $u = 0$ on $\partial\Omega$. From this we have some suitable multiple of u solves the desired equation and we are done.

We now show that (ii) holds for suitable large b. Let $-\Delta w = 1$ in Ω with $w = 0$ on $\partial\Omega$ and then we have $w \in K\backslash\{0\}$. Now we suppose there is some $b = R_k \to \infty$ and $\lambda_k \ge 0$ with $\|\nabla u_k\|_{L^\infty} = R_k$ and $u_k - T(u_k) = \lambda_k w$ and hence u_k solves

$$\begin{cases} -\Delta u_k = u_k^p + \lambda_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

Since $p > 1$ one can show that there must be some $C > 0$ such that $0 \le \lambda_k \le C$ for all k. Now the idea is to use a blow up argument to get a contradiction. If we suppose that $\|\nabla u_k\|_{L^\infty} \to \infty$ then we can apply standard elliptic theory to see that $\|u_k\|_{L^\infty} \to \infty$. By the monotonicity assumptions on u_k we have

$$\max_{\widehat{\Omega}} u_k = u_k(s_k, 0),$$

for some $g_1(0) < s_k < g_2(0)$ (ie. the max is attained the the s axis). We now perform a blow up argument around this point. Note in general we expect the limiting problem to be set in dimension $n+1$; the dimension n is coming from the t direction and $+1$ dimension is coming from the s direction which is bounded away from the origin. So one expects that after a blow up argument they will arrive at a nontrivial bounded solution of $-\Delta v = v^p$ in $\mathbb{R}^{n+1}$ or some related problem in a half space. To get the desired contradiction one now uses known Liouville theorems.

3. Proofs

Showing $T : K \to K$ is completely continuous. Showing $T(K) \subset K$ will follow from some general type arguments. For this we consider the case (4). Define

$$K = \left\{(u, v) \in C^1(\overline{\Omega}) \cap H^1_{0,G}(\Omega) : u, v \geq 0 \text{ and } u_\theta, v_\theta \leq 0 \text{ a.e. in } \widetilde{\Omega}\right\},$$

and T on K by $T(u, v) = (\widehat{u}, \widehat{v})$ where

$$\begin{cases} -\Delta \widehat{u} = u^{p_1} v^{q_1} & \text{in } \Omega, \\ -\Delta \widehat{v} = u^{p_2} v^{q_2} & \text{in } \Omega, \\ \widehat{u} = \widehat{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

To show that $(\widehat{u}, \widehat{v}) \in K$ we can apply the following lemma. Before we prove the lemma note to show T is completely continuous one can just apply standard elliptic regularity.

Lemma 1. *Suppose Ω is a monotonic domain of double revolution or an annular domain with monotonicity and suppose $f : [0, \infty) \to [0, \infty)$ is smooth and increasing. Let $u \in K$ (where K is defined by (18) and suppose v solves*

$$\begin{cases} -\Delta v = f(u) & \textit{in } \Omega, \\ v = 0 & \textit{on } \partial\Omega, \end{cases} \tag{23}$$

Then $v \in K$.

Proof. Let $u \in K$ we suppose v satisfies (23) and we can apply elliptic regularity to see that $v \in C^{2,\delta}(\overline{\Omega})$ some $\delta > 0$ and also note that v is just a function of (s, t). Since $v = v(s, t)$ is the restriction to the first quadrant of (x_1, x_{m+1}) plane of an even $C^{2,\delta}$ function in x_1 and x_{m+1} we see that $v_s, v_t \in C^{0,1}(\overline{\Omega})$ see [14] for this. This is sufficient regularity for v_s and v_t to give the desired boundary conditions on $\{s = 0\}$ and $\{t = 0\}$ portions of $\partial\widehat{\Omega}$ (we are assuming we are in the annular case and hence we don't need to worry about the origin; we will come back later and point out the needed adjustments for the non annular domains). In particular we have $v_s = 0$ on $\partial\widehat{\Omega} \cap \{s = 0\}$ and $v_t = 0$ on $\partial\widehat{\Omega} \cap \{t = 0\}$. Set $w = sv_t - tv_s = v_\theta$. Note that $w = 0$ on $\partial\widehat{\Omega} \cap (\{s = 0\} \cup \{t = 0\})$. We now show that $w \leq 0$ on the remainder of the boundary. To see this we first note that $w = v_\theta$ and since $v \geq 0$ and since g_2 is decreasing and g_1 is increasing we immediately see $v_\theta \leq 0$ on the remainder of the boundary after viewing $v = v(\theta, r)$ on $\widetilde{\Omega}$ (note without the monotonicity assumptions on g_i this is false). We now want to take a suitable derivatives of (23) to find an equation that w solves and use the maximum principle to show that $w \leq 0$ in $\widehat{\Omega}$. There are a few options here. We can either stay with the coordinates (s, t) or we can use polar coordinates (θ, r). Writing out (23) in terms of polar coordinates we arrive at

$$-v_{rr} - \frac{(N-1)v_r}{r} - \frac{v_{\theta\theta}}{r^2} + \frac{v_\theta}{r^2}\kappa(\theta) = f(u) \quad (\theta, r) \in \widetilde{\Omega},$$

where

$$\kappa(\theta) = (m-1)\tan(\theta) - \frac{(n-1)}{\tan(\theta)}. \tag{24}$$

Taking a derivative of this equation in θ we see that w solves

$$\widetilde{L}(w) := -w_{rr} - \frac{(N-1)w_r}{r} - \frac{w_{\theta\theta}}{r^2} + \kappa(\theta)\frac{w_\theta}{r^2} + \kappa'(\theta)\frac{w}{r^2} = f'(u)u_\theta, \quad (\theta, r) \in \widetilde{\Omega},$$

and note that

$$\kappa'(\theta) = \sec^2(\theta) \left\{ (m-1) + \frac{n-1}{\tan^2(\theta)} \right\},$$

and hence the zero order term has the correct sign for there to be hope to apply the maximum principle. Note we can rewrite $\widetilde{L}$ in terms of x coordinates to see that

$$\widetilde{L}(w) = -\Delta w(x) + \left\{ \frac{m-1}{x_1^2 + \cdots + x_m^2} + \frac{n-1}{x_{m+1}^2 + \cdots x_N^2} \right\} w(x),$$

and, up to issues with singularities, this again satisfies the maximum principle. Instead we will work in the (s,t) variables. A computation shows that

$$-w_{ss} - w_{tt} - \frac{(m-1)w_s}{s} - \frac{(n-1)w_t}{t} + \left\{ \frac{m-1}{s^2} + \frac{n-1}{t^2} \right\} w = h(s,t) \quad \text{in } \widehat{\Omega}, \tag{25}$$

where

$$h(s,t) = f'(u)(su_t - tu_s) \leq 0 \quad \text{in } \widehat{\Omega}.$$

Since $v_s, v_t \in C^{0,1}(\overline{\Omega})$ we see we have the same for w and hence we also have $w \in C^{0,1}(\overline{\widehat{\Omega}})$ (viewing w now as a function of (s,t)). Also recall we have $w \leq 0$ on $\partial\widehat{\Omega}$. For $\varepsilon > 0$ small we set $\psi(s,t) := (w(s,t) - \varepsilon)_+$ (the positive part of the function) and note $\psi = 0$ near $\partial\widehat{\Omega}$ and hence ψ is compactly supported in $\widehat{\Omega}$. Note in the case of a monotonic domain of double revolution where $(s,t) = (0,0) \in \partial\widehat{\Omega}$ we see that w is not defined at $(0,0)$. Note in this case we can use polar coordinates to estimate the gradient and we obtain the bound $|v_\theta| \leq Cr$ and hence we still have ψ (as defined above) is compactly supported in $\widehat{\Omega}$.

Test (25) on ψ (note the singularities in the equation are not an issue since ψ is compactly supported) to arrive at

$$\int_{\widehat{\Omega}} \nabla_{s,t} w \cdot \nabla_{s,t} \psi \, d\mu + \int_{\widehat{\Omega}} H(s,t) w \psi \, d\mu = \int_{\widehat{\Omega}} h \psi \, d\mu \leq 0,$$

where $H = \frac{m-1}{s^2} + \frac{n-1}{t^2}$ and recall $d\mu = s^{m-1} t^{n-1} ds dt$. From this we see that

$$\int_{\widehat{\Omega}} |\nabla_{s,t} (w - \varepsilon)_+|^2 d\mu + \int_{\widehat{\Omega}} H (w - \varepsilon)_+^2 d\mu \leq 0,$$

and hence we must have $(w - \varepsilon)_+ = 0$ a.e. (sense of μ) in $\widehat{\Omega}$ and so $w \leq \varepsilon$ a.e. $\widehat{\Omega}$ and since $\varepsilon > 0$ is arbitrary we have $w \leq 0$ a.e. in $\widehat{\Omega}$ and we are done. This shows that $v \in K$. $\square$

To solve any of the other problems listed one defines a suitable T and looks for fixed points on K. In all problems listed we will work on some version of K that involves nonnegative functions with the desired monotonicity in θ. In all these cases its fairly clear that Lemma 1 shows that $T(K) \subset K$.

Verifying part (i) of Theorem B. Again we consider the case of the general elliptic system (4) and we take T and K as defined by (22). In this case we use the norm $\|(u,v)\| := \|\nabla u\|_{L^\infty} + \|\nabla v\|_{L^\infty}$. We suppose (i) fails and hence there is some $\lambda > 1$ and $(u,v) \in K_a$ $(a > 0)$ with $T(u,v) = \lambda(u,v)$. Then (u,v) solves

$$-\Delta(\lambda u) = u^{p_1} v^{q_1}, \quad -\Delta(\lambda v) = u^{p_2} v^{q_2} \quad \text{in } \Omega,$$

with $u = v = 0$ on $\partial\Omega$. Provided $p_2 q_1 \neq (p_1 - 1)(q_2 - 1)$ then we can scale λ away (ie. take $\lambda = 1$) and we have found our positive solution. Otherwise we can always take a small to get the desired result. Suppose for all small $\varepsilon > 0$ there is some $\lambda_\varepsilon > 1$ and $\|(u_\varepsilon, v_\varepsilon)\| = \varepsilon$ with $T(u_\varepsilon, v_\varepsilon) = \lambda_\varepsilon(u_\varepsilon, v_\varepsilon)$. Putting $\widehat{u}_\varepsilon = \frac{u_\varepsilon}{\varepsilon}$ and $\widehat{v}_\varepsilon = \frac{v_\varepsilon}{\varepsilon}$ we see $\|(\widehat{u}_\varepsilon, \widehat{v}_\varepsilon)\| = 1$ and

$$-\Delta \widehat{u}_\varepsilon = \varepsilon^{p_1+q_1-1}\lambda_\varepsilon^{-1}(\widehat{u}_\varepsilon)^{p_1}(\widehat{v}_\varepsilon)^{q_1}, \quad -\Delta \widehat{v}_\varepsilon = \varepsilon^{p_2+q_2-1}\lambda_\varepsilon^{-1}(\widehat{u}_\varepsilon)^{p_2}(\widehat{v}_\varepsilon)^{q_2},$$

and since $p_1+q_1-1, p_2+q_2-1 > 0$ we can apply elliptic regularity theory to see that $\widehat{u}_\varepsilon, \widehat{v}_\varepsilon \to 0$ in $C^{0,1}(\overline{\Omega})$ as $\varepsilon \searrow 0$, a contradiction to the above normalization.

□

Verifying part (ii) of Theorem B. In all cases we will only verify part (ii) of Theorem B for large b. This will involve a blow up argument that we leave for Section 4. Again as a model case we consider (4) and the set up involving K and T as already defined. Let $-\Delta w = 1$ in Ω with $w = 0$ on $\partial\Omega$. Using the proof of Lemma 1 we see that $(w,w) \in K\backslash\{0\}$. We now suppose there is some $\lambda_k > 0$ and $(u_k, v_k) \in K_{R_k}$ (where $R_k \to \infty$) such that $(u_k, v_k) - T(u_k, v_k) = \lambda_k(w,w)$ and hence (u_k, v_k) satisfy

$$\begin{cases} -\Delta u_k = u_k^{p_1}v_k^{q_1} + \lambda_k & \text{in } \Omega, \\ -\Delta v_k = u_k^{p_2}v_k^{q_2} + \lambda_k & \text{in } \Omega, \\ u_k = v_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{26}$$

We now claim that λ_k is bounded above. Suppose $u_k, v_k > 0$ are classical solutions to (26). The maximum principle shows that $u_k, v_k \geq \lambda_k \tau$ where $-\Delta\tau = 1$ in Ω with $\tau = 0$ on $\partial\Omega$. If we multiply the first equation by $\frac{\phi^2}{u_k}$ and the second by $\frac{\phi^2}{v_k}$ (where $\phi \in C_c^\infty(\Omega)$) we can integrate by parts and add the resulting integrals to see

$$\lambda_k \int_\Omega \left(u_k^{-1} + v_k^{-1}\right)\phi^2 dx + \int_\Omega \left(u_k^{p_1-1}v_k^{q_1} + u_k^{p_2}v_k^{q_2-1}\right)\phi^2 dx \leq 2\int_\Omega |\nabla\phi|^2 dx.$$

Dropping the first integral and putting this bounds for u_k, v_k gives

$$\int_\Omega \left(\lambda_k^{p_1+q_1-1}\tau^{p_1+q_1-1} + \lambda_k^{p_2+q_2-1}\tau^{p_2+q_2-1}\right)\phi^2 dx \leq 2\int_\Omega |\nabla\phi|^2 dx.$$

From this we can conclude that $\{\lambda_k\}_k$ is bounded above. To complete the proof one needs to perform a blow up argument on (26) and we save this for the next section.

The Lane-Emden system (5). This system does not fit exactly into the framework we used to show that $\{\lambda_k\}_k$ is bounded. So we suppose $(u_,v_k)$ is a sequence of positive classical solutions of

$$\begin{cases} -\Delta u_k = v_k^p + \lambda_k & \text{in } \Omega, \\ -\Delta v_k = u_k^q + \lambda_k & \text{in } \Omega, \\ u_k = v_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

Let $-\Delta\phi = \mu\phi$ in Ω with $0 \in H_0^1(\Omega)$ (the first eigenpair of $-\Delta$ in $H_0^1(\Omega)$. Without loss of generality we can assume

$$\int_\Omega u_k^q \phi dx \leq \int_\Omega v_k^p \phi dx.$$

Multiply the first equation of (27) by ϕ and integrate to arrive at

$$\begin{aligned}\lambda_k \int_\Omega \phi dx + \int_\Omega v_k^p \phi dx &= \mu \int_\Omega u_k \phi dx \\ &= \mu \int_\Omega (u_k \phi^{\frac{1}{q}}) \phi^{\frac{1}{q'}} dx \\ &\le \mu\varepsilon \int_\Omega u_k^q \phi dx + \mu C(\varepsilon, q) \int_\Omega \phi dx \\ &\le \mu\varepsilon \int_\Omega v_k^p \phi dx + \mu C(\varepsilon, q) \int_\Omega \phi dx,\end{aligned}$$

and now by taking $\varepsilon > 0$ small enough and then grouping integrals on the left we see that $\lambda_k \int_\Omega \phi dx \le C$ and hence we have $\{\lambda_k\}_k$ is bounded. □

4. The blow up arguments

4.1. **The Hénon equation.** To verify part (ii) of Theorem B we take K as defined in (18) and T as defined in the analougous way to (19)). Set w such that $-\Delta w = |x|^\alpha$ in Ω with $w = 0$ on $\partial\Omega$ and one can show that $w \in K\backslash\{0\}$. Towards a contradiction we suppose $\|\nabla u_k\| \to \infty$ is such that $u_k - T(u_k) = \lambda_k w$ and hence u_k satisfies

$$\begin{cases} -\Delta u_k = |x|^\alpha u_k^p + \lambda_k |x|^\alpha & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{28}$$

Using a similar argument as in the system we can show that λ_k is bounded. Since $\|\nabla u_k\|_{L^\infty} \to \infty$ we can show that $T_k := \sup_\Omega u_k \to \infty$ by standard elliptic regularity. We now perform a blow up argument to obtain the needed contradiction. Now recall we are in a monotonic domain of double of revolution; so the origin does belong to Ω. Using the monotonicity of u_k we see that $T_k = u^k(s_k, 0)$ for some $0 \le s_k < g(0)$. For $r_k > 0$ define

$$v^k(s,t) = \frac{u^k(s_k + r_k s, r_k t)}{T_k} \quad \text{for } (s,t) \in \widehat{\Omega}_k := \left\{(s,t) : (s_k + r_k s, r_k t) \in \widehat{\Omega}\right\},$$

and then a computation shows that

$$-v^k_{ss} - v^k_{tt} - \frac{(m-1) r_k v^k_s}{s_k + r_k s} - \frac{(n-1) v^k_t}{t} = I_k(s,t) \left\{ (v^k)^p + \frac{\lambda_k}{T_k^p} \right\} \text{ on } (s,t) \in \widehat{\Omega}_k, \tag{29}$$

where

$$I_k(s,t) = T_k^{p-1} r_k^2 \left((s_k + r_k s)^2 + r_k^2 t^2 \right)^{\frac{\alpha}{2}},$$

with $v^k_t = 0$ on $\partial\widehat{\Omega}_k \cap \{t = 0\}$ and $v^k_s = 0$ on $\partial\widehat{\Omega}_k \cap \{s = \frac{-s_k}{r_k}\}$ and $v^k = 0$ on the remainder of the boundary.

We now consider three cases:
(I) $T_k^{p-1} s_k^{2+\alpha} \to 0$, (II) $T_k^{p-1} s_k^{2+\alpha} \to \gamma \in (0, \infty)$, (III) $T_k^{p-1} s_k^{2+\alpha} \to \infty$.

Case (I). We assume $s_k > 0$; the case of $s_k = 0$ is even easier. Alternatively we could find an $s_k > 0$ such that $u^k(s_k, 0) \ge \frac{9T_k}{10}$ and then proceed and obtain the same contradiction. In this case define $\varepsilon_k^{2+\alpha} = T_k^{p-1} s_k^{2+\alpha} \to 0$ and define r_k by $s_k = \varepsilon_k r_k$. Then note we have $1 = T_k^{p-1} r_k^{2+\alpha}$ and hence $r_k \to 0$. Also note

$$I_k(s,t) = ((\varepsilon_k + s)^2 + t^2)^{\frac{\alpha}{2}} \to (s^2 + t^2)^{\frac{\alpha}{2}}.$$

Also note that
$$\frac{r_k}{s_k + r_k s} \to \frac{1}{s}.$$
Note then, at least formally, we expect a subsequence to converge in say $C^{1,\delta}$ to some v with $v(0,0) = 1$ and $0 \le v \le 1$ a solution of
$$-v_{ss} - v_{tt} - \frac{(m-1)v_s}{s} - \frac{(n-1)v_t}{t} = (s^2+t^2)^{\frac{\alpha}{2}} v^p \text{ in } \widehat{\Omega}_\infty,$$
where the domain is some limiting domain. Fix $\delta > 0$ small such that $Q := (0,\delta)^2 \subset \widehat{\Omega}$ and set $\widehat{Q}_k := \{(s,t) : (s_k + r_k s, r_k t) \in Q\}$ and note $\widehat{Q}_k \subset \widehat{\Omega}_k$. Then note that
$$\widehat{Q}_k = \left(-\varepsilon_k, \frac{\delta - s_k}{r_k}\right) \times \left(0, \frac{\delta}{r_k}\right),$$
and so note that $\widehat{Q}_k \to (0,\infty)^2$. Also note that the lower and left boundary of $\widehat{Q}_k$ is also a portion of the boundary of $\widehat{\Omega}_k$. Note also that v^k attains in maximum on the boundary and so to pass to the limit we need to be a bit careful. One approach would be to use boundary regularity. Another option is to extend v_k evenly across the left and lower boundary. This then allows one to use interior regularity and to pass to a limit to obtain a nonzero solution (written in terms of x) v of
$$-\Delta v(x) = |x|^\alpha v(x)^p \text{ in } \mathbb{R}^N,$$
but this contradicts [38, 68], see Section 5.

Case (II). In this case take $r_k = s_k$ and hence $r_k \to 0$. Then
$$I_k(s,t) = T_k^{p-1} s_k^{2+\alpha} ((1+s)^2 + t^2)^{\frac{\alpha}{2}},$$
and
$$\frac{r_k}{s_k + r_k s} = \frac{1}{1+s}.$$
Again we can pass to a limit to find some v with $v(0,0) = 1$ and $0 \le v \le 1$ which satisfies
$$-v_{ss} - v_{tt} - \frac{(m-1)v_s}{1+s} - \frac{(n-1)v_t}{t} = \gamma \left((1+s)^2 + t^2\right)^{\frac{\alpha}{2}} v^p \text{ in } \widehat{\Omega}_\infty,$$
where
$$\widehat{\Omega}_\infty = (-1,\infty) \times (0,\infty),$$
with the appropriate Neumann boundary conditions. Note this is the same limiting equation that was obtained in case (I) but there is a shift in s and also the maximum is attained at a different location. Again we can obtain the needed contradiction by [38, 68] as in the previous case.

Case (III). Set $\varepsilon_k > 0$ such that $\varepsilon_k^2 := T_k^{p-1} s_k^{2+\alpha} \to \infty$ and set $r_k = \frac{s_k}{\varepsilon_k}$ and hence note $1 = T_k^{p-1} \varepsilon_k^\alpha r_k^{2+\alpha}$ and hence $r_k \to 0$. In this case note that
$$I_k(s,t) = \left((1 + \frac{s}{\varepsilon_k})^2 + \frac{t^2}{\varepsilon_k^2}\right)^{\frac{\alpha}{2}} \to 1,$$
and
$$\frac{r_k}{s_k + r_k s} = \frac{1}{\varepsilon_k + s} \to 0.$$

In this case we really need to consider three subcases:
(i) $\frac{g(0)-s_k}{r_k} \to \infty$, (ii) $\frac{g(0)-s_k}{r_k} \to 0$, (iii) $\frac{g(0)-s_k}{r_k} \to \gamma \in (0,\infty)$.

Note that case (III) allows the three cases of $s_k \to 0$ slowly, s_k bounded away from zero and $g(0)$ and also the case where s_k approaches $g(0)$.

Note case (i) includes the above mentioned cases restricts s_k convergence to $g(0)$ to be slowly. Case (ii) we have $s_k \to g(0)$ fast and case (iii) we have intermediate speed of convergence.

Case (i). We first consider the cases where s_k is bounded away from $g(0)$ (which includes the case of $s_k \to 0$ slowly). Then there is some $\delta > 0$ (small) such that

$$Q_k = \left(\frac{s_k}{2}, s_k + \delta\right) \times (0,\delta) \subset \widehat{\Omega},$$

for all k large and we set

$$\widehat{Q}_k := \left\{(s,t) : (s_k + r_k s, r_k t) \in Q_k\right\},$$

and note $\widehat{Q}_k \subset \widehat{\Omega}_k$, and note that

$$\widehat{Q}_k = \left(\frac{-\varepsilon_k}{2}, \frac{\delta}{r_k}\right) \times \left(0, \frac{\delta}{r_k}\right),$$

and recall that $r_k \to 0$ and $\varepsilon_k \to \infty$. So note that $\widehat{Q}_k \to \mathbb{R} \times (0,\infty)$. In this case we can again extend evenly in t to help one pass to the limit. In any case $v^k \to v$ where v is nonzero and satisfies

$$-v_{ss} - v_{tt} - \frac{(n-1)v_t}{t} = v^p \quad \text{ on } (s,t) \in \mathbb{R} \times (0,\infty),$$

with $v_t = 0$ on the boundary. Note the dimension here is $n+1$ (the t variable is giving dimension n and the s variable is only adding one extra dimension). In terms of x we have

$$-\Delta v(x) = v(x)^p \quad \text{ in } \mathbb{R}^{n+1},$$

and we get the desired contradiction provided $p < \frac{(n+1)+2}{(n+1)-2}$ from the well known classical Liouville theorem.

We now assume we are in the case of $s_k \to g(0)$ but with $\frac{g(0)-s_k}{r_k} \to \infty$ (so slow convergence to the outer boundary). In this case consider

$$Q_k = \left(\frac{3s_k - g(0)}{2}, \frac{g(0)+s_k}{2}\right) \times \left(0, \delta(g(0)-s_k)\right),$$

where $\delta > 0$ is chosen small and fixed (independently of k) and we claim that $Q_k \subset \widehat{\Omega}$ for large enough k. To see this write $s = h(t)$ with $h(0) = g(0)$ and $h'(0) = 0$ and h smooth; we omit the details. Set $\widehat{Q}_k = \{(s,t) : (s_k + r_k s, r_k t) \in Q_k\} \subset \widehat{\Omega}_k$ and note

$$\widehat{Q}_k = \left(\frac{-(g(0)-s_k)}{2r_k}, \frac{g(0)-s_k}{2r_k}\right) \times \left(0, \frac{\delta(g(0)-s_k)}{r_k}\right) \to \mathbb{R} \times (0,\infty).$$

In this case we get the same limiting equation and domain and hence we again obtain our desired contradiction.

Case (iii). $\frac{g(0)-s_k}{r_k} = \gamma_k \to \gamma \in (0,\infty)$. In this case one sees that $\widehat{\Omega}_k \to \widehat{\Omega}_\infty = (-\infty,\gamma) \times (0,\infty)$. So passing to the limit we find a nonnegative nonzero bounded solution of

$$-v_{ss} - v_{tt} - \frac{(n-1)v_t}{t} = v^p \quad \text{ on } (s,t) \in (-\infty,\gamma) \times (0,\infty),$$

with $v_t = 0$ on the bottom boundary and $v = 0$ on the right boundary. We can now extend evenly in t across $t = 0$ to find a solution (we still write it as v) of $-v_{ss} - v_{tt} - \frac{(n-1)v_t}{t} = v^p$ on $(s,t) \in (-\infty,\gamma) \times \mathbb{R}$ with $v(\gamma,t) = 0$. Note that since s is one dimension and t is n dimension we see that we have found a nonzero nonnegative solution of $-\Delta v = v^p$ in $\mathbb{R}^{n+1}_+$ with with zero Dirichlet boundary condition which contradicts some half space Liouville theorems; see [36, 37].

Case (ii). $\frac{g(0)-s_k}{r_k} = \gamma_k \to 0$. In this case we will redefine r_k and so note we can rewrite this case (iii) as

$$\gamma_k = (g(0) - s_k) T_k^{\frac{p-1}{2}} s_k^{\frac{\alpha}{2}} \to 0,$$

and note the s_k is not playing a role since its bounded away from zero. We will now take $r_k = g(0) - s_k$ which goes to zero and now we will rewrite (29). So note we have $\frac{r_k}{s_k + r_k s} \to 0$ and $I_k(s,t) \to 0$. Passing to the limit in (29) we arrive at some nonnegative nonzero v which satisfies

$$-v_{ss} - v_{tt} - \frac{(n-1)v_t}{t} = 0 \quad (s,t) \in (-\infty,1) \times (0,\infty),$$

with $v_t = 0$ on the bottom boundary and $v = 0$ on the right boundary and also we have $0 \le v \le 1$ with $v(0,0) = 1$. We can now extend evenly in t and then get a contradiction to the strong maximum principle after considering the origin. This completes the proof of the apriori estimates for the Hénon equation. □

4.2. **The case of the elliptic systems.** Here we consider (4). Suppose (u^k, v^k) is sequence of positive classical solutions of (26) with λ_k bounded and we suppose $\|(u^k,v^k)\| = \|\nabla u^k\|_{L^\infty} + \|\nabla v^k\|_{L^\infty} \to \infty$. By standard elliptic estimates we see that $\sup_\Omega (u_k + v_k) \to \infty$. We suppose p_i, q_i are such that there is a solution (α,β) of

$$(p_1 - 1)\alpha + q_1\beta = 2, \quad p_2\alpha + (q_2 - 1)\beta = 2, \qquad \alpha,\beta > 0. \tag{30}$$

Without loss of generality we can suppose

$$\|u^k\|_{L^\infty}^{\frac{1}{\alpha}} \ge \|v^k\|_{L^\infty}^{\frac{1}{\beta}}.$$

We now set

$$r_k := \frac{1}{\|u^k\|_{L^\infty}^{\frac{1}{\alpha}} + \|v^k\|_{L^\infty}^{\frac{1}{\beta}}},$$

and note $r_k \to 0$. We now work in (s,t) coordinates in $\widehat{\Omega}$. By monotonicity in θ we see there is some $(s_k, 0)$ with $g_1(0) < s_k < g_2(0)$ such $\|u^k\|_{L^\infty} = u^k(s_k, 0)$. Now consider

$$\widehat{u}^k(s,t) := r_k^\alpha u^k(s_k + r_k s, r_k t), \quad \widehat{v}^k(s,t) := r_k^\beta v^k(s_k + r_k s, r_k t),$$

for

$$(s,t) \in \widehat{\Omega}_k := \left\{ (s,t) : (s_k + r_k s, r_k t) \in \widehat{\Omega} \right\}.$$

A computation shows that $0 \leq \widehat{u}^k, \widehat{v}^k \leq 1$ with $\widehat{u}^k(0,0) \geq 2^{-\alpha}$. Note that (α, β) are such that these scaled solutions satisfy the same equation but with slightly different λ_k. In terms of the (s,t) coordinates we have $\widehat{u}^k(s,t)$ and $\widehat{v}^k(s,t)$ satisfy

$$-\widehat{u}^k_{ss} - \widehat{u}^k_{tt} - \frac{(m-1)r_k}{s_k + r_k s}\widehat{u}^k_s - \frac{(n-1)\widehat{u}^k_t}{t} = (\widehat{u}^k)^{p_1}(\widehat{v}^k)^{q_1} + \lambda_k r_k^{\alpha+2}, \tag{31}$$

$$-\widehat{v}^k_{ss} - \widehat{v}^k_{tt} - \frac{(m-1)r_k}{s_k + r_k s}\widehat{v}^k_s - \frac{(n-1)\widehat{v}^k_t}{t} = (\widehat{u}^k)^{p_2}(\widehat{v}^k)^{q_2} + \lambda_k r_k^{\beta+2}, \tag{32}$$

for $(s,t) \in \widehat{\Omega}^k$.

Define

$$\Gamma_1^k := \left\{ (s,0) : \frac{g_1(0) - s_k}{r_k} < s < \frac{g_2(0) - s_k}{r_k} \right\},$$

and we set $\partial\widehat{\Omega}_{out}$, $\partial\widehat{\Omega}_{in}$ to denote the outer and inner portion of the boundary of $\widehat{\Omega}$ (the portions not on the s or t axis). Then define

$$\Gamma_{out}^k := \left\{ (s,t) : (s_k + r_k s, r_k t) \in \partial\widehat{\Omega}_{out} \right\},$$

$$\Gamma_{in}^k := \left\{ (s,t) : (s_k + r_k s, r_k t) \in \partial\widehat{\Omega}_{in} \right\}.$$

Then note we have $\widehat{u}^k_t = 0$ on Γ_1^k and $\widehat{u}^k = 0$ on $\Gamma_{out}^k \cup \Gamma_{in}^k$ and similarly for $\widehat{v}^k$. Note we also have a Neumann boundary condition on the remaining portion of the boundary but since we blowing up around the point $(s_k, 0)$ we will not see the other portion of the boundary.

There are multiple cases we need to consider during the blow up.

$$(I)\ \frac{g_1(0) - s_k}{r_k} \to -\infty \ \text{ and } \ \frac{g_2(0) - s_k}{r_k} \to \infty,$$

$$(II)\ \frac{s_k - g_1(0)}{r_k} = \gamma_k \to \gamma \in [0, \infty),$$

and a similar case where s_k approaches $g_2(0)$ in a sufficiently fast way. This last case will follow with exactly the same arguments as in case (II) so we will omit it.

Case (I). In this case we can pass to the limit in (31) and (32) to find a classical positive bounded nonzero solution of

$$-\widehat{u}_{ss} - \widehat{u}_{tt} - \frac{(n-1)\widehat{u}_t}{t} = (\widehat{u})^{p_1}(\widehat{v}^k)^{q_1}, \tag{33}$$

$$-\widehat{v}_{ss} - \widehat{v}_{tt} - \frac{(n-1)\widehat{v}_t}{t} = (\widehat{u})^{p_2}(\widehat{v}^k)^{q_2}, \tag{34}$$

in $(s,t) \in \mathbb{R} \times (0,\infty)$ with $u_t = v_t = 0$ on $t = 0$. To pass to the limit one can first extend evenly in t before passing to the limit. In either case in terms of variable x one has a nonzero nonnegative solution

$$-\Delta\widehat{u} = \widehat{u}^{p_1}\widehat{v}^{q_1}, \quad -\Delta\widehat{v} = \widehat{u}^{p_2}\widehat{v}^{p_2} \quad \text{ in } \mathbb{R}^{n+1}.$$

One needs a suitable Liouville theorem to rule this case out.

Case (II). In this case $\widehat{\Omega}^k \to \widehat{\Omega}^\infty = \{(s,t) : s > -\gamma, t > 0\}$. We first consider the case of $\gamma > 0$. In this case we can pass to a limit to find a nonzero nonnegative classical bounded solution of note satisfy (33) and (34) on $\widehat{\Omega}^\infty$ with $\widehat{u}(-\gamma, t) =$

$\widehat{v}(-\gamma,t)=0$ for all $t>0$. Additionally they satisfy $\widehat{u}_t(s,0)=\widehat{v}_t(s,0)=0$ for $s>-\gamma$. After translating by γ in the s directly we can view this as a nonnegative nonzero solution of

$$-\Delta\widehat{u}=\widehat{u}^{p_1}\widehat{v}^{q_1}, \quad -\Delta\widehat{v}=\widehat{u}^{p_2}\widehat{v}^{p_2} \quad \text{in } \mathbb{R}^{n+1}_+,$$

with $\widehat{u}=\widehat{v}=0$ on $\partial\mathbb{R}^{n+1}_+$.

We now suppose $\gamma=0$. In this case we will use a slightly different scaling. We still define r_k as before and then define

$$\widehat{u}^k(s,t):=r_k^\alpha u^k(s_k+r_k\gamma_k s, r_k\gamma_k t), \quad \widehat{v}^k(s,t):=r_k^\beta v^k(s_k+r_k\gamma_k s, r_k\gamma_k t),$$

for

$$(s,t)\in\widehat{\Omega}_k:=\left\{(s,t):(s_k+r_k\gamma_k s, r_k\gamma_k t)\in\widehat{\Omega}\right\}.$$

As before we still have $0\le\widehat{u}^k,\widehat{v}^k\le 1$ with $\widehat{u}^k(0,0)\ge 2^{-\alpha}$. In terms of the (s,t) coordinates we have $\widehat{u}^k(s,t)$ and $\widehat{v}^k(s,t)$ satisfy

$$-\widehat{u}^k_{ss}-\widehat{u}^k_{tt}-\frac{(m-1)r_k\gamma_k}{s_k+r_k\gamma_k s}\widehat{u}^k_s-\frac{(n-1)\widehat{u}^k_t}{t}=\varepsilon_k(\widehat{u}^k)^{p_1}(\widehat{v}^k)^{q_1}+\lambda_k r_k^{\alpha+2}\gamma_k^2, \tag{35}$$

$$-\widehat{v}^k_{ss}-\widehat{v}^k_{tt}-\frac{(m-1)r_k\gamma_k}{s_k+r_k\gamma_k s}\widehat{v}^k_s-\frac{(n-1)\widehat{v}^k_t}{t}=\delta_k(\widehat{u}^k)^{p_2}(\widehat{v}^k)^{q_2}+\lambda_k r_k^{\beta+2}\gamma_k^2, \tag{36}$$

for $(s,t)\in\widehat{\Omega}^k$ where ε_k,δ_k are positive constants which converge to zero. Note in this case we have that $\widehat{\Omega}\to\widehat{\Omega}^\infty=\{(s,t):s>-1,t>0\}$. By passing to a limit in the equation we find nonnegative, bounded nonzero solution $(\widehat{u},\widehat{v})$ of

$$\widehat{u}_{ss}+\widehat{u}_{tt}+\frac{(n-1)\widehat{u}_t}{t}=0, \widehat{v}_{ss}+\widehat{v}_{tt}+\frac{(n-1)\widehat{v}_t}{t}=0, \quad \text{in } \widehat{\Omega}^\infty,$$

with $\widehat{u}(-1,t)=\widehat{v}(-1,t)=0$ for all $t>0$ and $\widehat{u}_t(s,0)=\widehat{v}_t(s,0)=0$ for all $s>-1$. Also note that $0\le\widehat{u}\le 1$ with $u(0,0)\ne 0$. By extending evenly in t then we have a bounded nonnegative nonzero harmonic function $\widehat{u}$ in a shifted halfspace. Then we must have $\widehat{u}$ is constant by known classical Liouville theorems and hence we must have $\widehat{u}=0$ after considering the boundary condition, but this contradicts the fact that $\widehat{u}(0,0)\ne 0$.

5. Liouville theorems

Here we collect some Liouville theorems we will apply in the current work. Note we are not attempting to be exhaustive at all here. We only list results for the Hénon equation and also the Lane-Emden system.

(1) (Hénon equation). The only nonnegative classical solution of $-\Delta u(x)=|x|^\alpha u(x)^p$ in $\mathbb{R}^N$ for $1<p<p_\alpha(N)$ is $u=0$, see [38] and [68]. This optimal result was only known in dimension $N=3$ until these recent works.

(2) (A Lane-Emden system) The Lane-Emden Conjecture states the only nonnegative bounded classical solution of

$$-\Delta u=v^p, -\Delta v=u^q \quad \text{in } \mathbb{R}^N,$$

is $u=v=0$ provided

$$\frac{1}{p+1}+\frac{1}{q+1}>1-\frac{2}{N}.$$

This conjecture has only been proven in dimension $N \leq 4$, see [67]. In higher dimensions there are only partial results. We also need Liouville theorems for half spaces. Sufficient half spaces theorems are available for this case; see [24]. By sufficient we mean that we do not need optimal half space results but rather just sufficiently good results that at least have the same parameter range as the available theorems for the full space.

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3. I am the sole author and hence I did all the work in the paper.

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Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2

Email address: craig.cowan@umanitoba.ca