Perturbations of Lane-Emden and Hamilton-Jacobi equations: the full space case

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Abstract

In this article we are interested in positive classical solutions of

$$-\Delta u + a(x) \cdot \nabla u + V(x) u = u^p + \gamma u^q \qquad \text{in } \mathbb{R}^N,$$

and

$$-\Delta u + a(x) \cdot \nabla u = u^p + \gamma |\nabla u|^q \qquad \text{in } \mathbb{R}^N,$$

in the case of $N \geq 4$, $p > \frac{N+1}{N-3}$ and $\gamma \in \mathbb{R}$. We assume that V is a smooth non-negative potential and $a(x)$ is a smooth vector field, both of which satisfy natural decay assumptions. Under suitable assumptions on q we prove the existence of positive classical solutions.

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1 Introduction and statement of main results

In this article we are interested in positive classical solutions of the following variants of the Lane-Emden and viscous Hamilton-Jacobi equations given by

$$-\Delta u + a(x) \cdot \nabla u + V(x) u = u^p + \gamma u^q \qquad \text{in } \mathbb{R}^N, \tag{1}$$

and

$$-\Delta u + a(x) \cdot \nabla u = u^p + \gamma |\nabla u|^q \qquad \text{in } \mathbb{R}^N, \tag{2}$$

where $p, q > 1$, $\gamma \in \mathbb{R}$ and

$(A1):$ $a(x)$ is a smooth vector field satisfing $\lim_{R\to\infty} A(R) = 0$ where $A(R) := \sup_{|x|\geq R} |x||a(x)|$,

$(A2):$ $V(x) \geq 0$ is a smooth potential satisfing $\lim_{R\to\infty} \tilde{V}(R) = 0$ where $\tilde{V}(R) := \sup_{|x|\geq R} |x|^2 V(x)$.

Throughout this work we take $\alpha := \frac{2}{p-1}$. We begin by recalling the bounded domain version of (1) in the case of $a(x) = 0, V(x) = 0$ and $\gamma = 0$ given by

$$\left\{ \begin{array}{rcll} -\Delta u &=& u^p & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 3$. Define the critical exponent $p_s = \frac{N+2}{N-2}$ and note that it is related to the critical Sobolev imbedding exponent $2^* := \frac{2N}{N-2} = p_s + 1$. For $1 < p < p_s$ $H_0^1(\Omega)$ is compactly imbedded in $L^{p+1}(\Omega)$ and hence standard methods show the existence of a positive minimizer of

$$\min_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}}.$$

This positive minimizer is a positive solution of (3) see for instance the book [19]. For $p \geq p_s$ $H_0^1(\Omega)$ is no longer compactly imbedded in $L^{p+1}(\Omega)$ and so to find positive solutions of (3) one needs to take other approach. For $p \geq p_s$ the well known Pohozaev identity [18] shows there are no positive solutions of (3) provided Ω is star shaped. For general domains in the critical/supercritical case, $p \geq p_s$, the existence versus nonexistence of positive solutions of (3) is a very delicate question; see [5, 7, 17].

We now recall (1) in the case of $a(x) = 0$, $V(x) = 0$ and $\gamma = 0$. There has been much work done on the existence and nonexistence of positive classical solutions of

$$-\Delta w = w^p \qquad \text{in } \mathbb{R}^N. \tag{4}$$

As in the bounded domain case the critical exponent p_s plays a crucial role. For $1 < p < p_s$ there are no positive classical solutions of (4) and for $p \geq p_s$ there exist positive classical solutions, see [3, 4, 13, 12]. The moving plane method shows that all positive classical solutions, satisfying certain assumptions, are radial about a point. In [6] the existence versus nonexistence of stable positive solutions of $-\Delta u + a(x) \cdot \nabla u = u^p$ in $\mathbb{R}^N$ was considered. Results were obtained that depended on smallness assumptions on a and removing this smallness assumption was the motivation for the current work. The interested reader should consult [21, 14] for this question in the case of $a = 0$.

In the current work our approach to finding positive classical solutions of (1) and (2) are motivated by the approach from Dávila-del Pino-Musso-Wei [10]. In [10] they examined equations of the form $-\Delta u(x) + V(x)u(x) = u(x)^p$ in $\mathbb{R}^N$ and they treated the equation as a perturbation of the pure power problem (4). To solve the perturbed problem they first needed a detailed study of the linearized operator associated with (4) given by $L(\phi) := \Delta\phi + pw^{p-1}\phi$, where w is the positive radial solution of (4) with $w(0) = 1$ (see below for the asymptotics of w). This analysis had already been carried out on the weighted spaces X_0 and Y_2 in Dávila-del Pino-Musso [8] where they obtained positive solutions of (4) on exterior domains. We now state their exact linear theory which requires us to define some Banach spaces on the punctured domain $\mathbb{R}^N \backslash \{0\}$: for $\sigma > 0$ define $X_0 := \{\phi \in C(\mathbb{R}^N \backslash \{0\}) : \|\phi\|_{X_0} < \infty\}$ where

$$\|\phi\|_{X_0} := \sup_{0<|x|\leq 1} |x|^\sigma |\phi(x)| + \sup_{|x|\geq 1} |x|^\alpha |\phi(x)|;$$

$Y_2 := \{f \in C(\mathbb{R}^N \backslash \{0\}) : \|f\|_{Y_2} < \infty\}$ where

$$\|f\|_{Y_2} := \sup_{0<|x|\leq 1} |x|^{\sigma+2} |f(x)| + \sup_{|x|\geq 1} |x|^{\alpha+2} |f(x)|.$$

Theorem A. *[8, 10] Suppose $N \geq 4$ and $p > \frac{N+1}{N-3}$. Then for all $0 < \sigma < N-2$ there is some $C > 0$ such that for every $f \in Y_2$ there exists some $\phi \in X_0$ such that $L(\phi) = f$ in $\mathbb{R}^N \backslash \{0\}$ and $\|\phi\|_{X_0} \leq C \|f\|_{Y_2}$.*

Asymptotics of w. The asymptotics of w are

$$w(r) = \beta^{\frac{1}{p-1}} r^{\frac{-2}{p-1}} (1 + o(1)) \quad \text{as } r \to \infty,$$

where

$$\beta = \beta(p, N) = \frac{2}{p-1} \left(N - 2 - \frac{2}{p-1} \right) > 0,$$

see [14] for this and for more detailed asymptotics.
As mentioned earlier (1) was examined in [6] in the case of $\gamma = 0$ and $V(x) = 0$ under a smallness assumption on $a(x)$. This work involved an existence portion where the above linear theory needed to be extended to a slightly different Banach space: $X_1 := \{\phi \in C^1(\mathbb{R}^N \backslash \{0\}) : \|\phi\|_{X_1} < \infty\}$ where

$$\begin{aligned} \|\phi\|_{X_1} \quad := \quad & \sup_{0<|x|\leq 1} \left(|x|^\sigma |\phi(x)| + |x|^{\sigma+1} |\nabla \phi(x)| \right) \\ & + \sup_{|x|\geq 1} \left(|x|^\alpha |\phi(x)| + |x|^{\alpha+1} |\nabla \phi(x)| \right). \end{aligned}$$

Corollary A. *[6] Suppose $N \geq 4$ and $p > \frac{N+1}{N-3}$. For $0 < \sigma < N-2$ there is some $C > 0$ such that for every $f \in Y_2$ there exists some $\phi \in X_1$ such that $L(\phi) = f$ in $\mathbb{R}^N \backslash \{0\}$ and $\|\phi\|_{X_1} \leq C \|f\|_{Y_2}$.*

1.1 Statement of main results

We now state our results. Our first theorem is with regards to (1).

Theorem 1. *Suppose $N \geq 4$, $q > p > \frac{N+1}{N-3}$ and $(A1), (A2)$ are satisfied.*

1. *Suppose $\gamma \geq 0$. Then there is a smooth positive solution u of (1).*

2. *Suppose* $\gamma < 0$ *and*

$$\|(div(a) - 2V)_+\|_{L^{\frac{N}{2}}(\mathbb{R}^N)} < 2S_N,$$

where $(div(a) - 2V)_+$ *is the positive part of* $div(a) - 2V$ *and* S_N *is the optimal constant in the critical Sobolev imbedding, see Lemma 5. Then there is a smooth positive solution* u *of (1).*

Note that $\|(div(a) - 2V)_+\|_{L^{\frac{N}{2}}(\mathbb{R}^N)} < 2S_N$ is trivially satisfied if $a(x)$ is divergence free. We now consider (2) for which we obtain various results; each result corresponds to a fixed point argument on a different space. We feel the most natural approach to take when considering (2) is given by the approach we take in part 4 of Theorem 2. This approach relies on a linear Liouville theorem that we suspect should hold but we have not managed to prove it so we add the needed condition to the hypothesis.

Theorem 2. *Suppose* $\gamma \in \mathbb{R}$, $N \geq 4$, $p > \frac{N+1}{N-3}$ *and* $q > \frac{2p}{p+1}$.

1. *Suppose* $a(x) = 0$. *Then there exists a positive classical solution of (2).*

2. *Suppose* $a(x)$ *satisfies* $(A1)$ *and* $q < 2$. *Then there is a positive classical solution of (2).*

3. *Suppose* $a(x)$ *is divergence free and satisfies* $(A1)$ *and suppose* $q \geq p$. *Then there is a positive classical solution of (2).*

4. *Suppose* $a(x)$ *is divergence free and satisfies* $(A1)$ *and suppose for all* $0 < \sigma < 1$ *the only smooth solutions* ψ *of* $\Delta\psi(x) - a(x) \cdot \nabla\psi(x) = 0$ *in* $\mathbb{R}^N$, *which satisfies* $|\nabla\psi(x)| \leq \frac{C}{|x|^\sigma}$, *are the constant solutions (see Remark 1 part (c) for comments on this assumption). Then there is a positive classical solution of (2).*

We now state our final result regarding (2). In the first two theorems we work on various function spaces and in all cases we take the parameter $\sigma > 0$ small (which relates to allowable blow up at the origin) when we apply Banach's fixed point theorem. In our final result we don't take $\sigma > 0$ small. Doing this allows us to gain a range of allowable q when solving (2). Here we only consider the case of $q > 2$ since the case of $q < 2$ is handled in the previous theorem. We will obtain positive solutions of (2) under the assumption that

$$\frac{2}{p-1}(q-1)^2 - (q-1) + 1 > 0. \tag{5}$$

We now examine this condition in some detail. For $p < 9$, (5) is satisfied for all q. For $p > 9$, (5) has two zeros: $q_- < q_+$, which are explicitly given by

$$q_\pm := 1 + \frac{p-1}{4}\left(1 \pm \sqrt{1 - \frac{8}{p-1}}\right).$$

We can now state our final result.

Theorem 3. *Suppose* $q > 2$, $\gamma \in \mathbb{R}$, $N \geq 4$, $p > \frac{N+1}{N-3}$ *and* a *is divergence free and satisfies* $(A1)$.

1. *Suppose* $p < 9$. *Then there is a positive classical solution of (2).*

2. *Suppose* $p > 9$ *and* $2 < q < q_-$ *or* $q > q_+$. *Then there is a positive classical solution of (2).*

Remark 1. *(a) For presentation purposes we decided to not include the term* $V(x)u$ *in (2). But the same methods can easily be applied to extend Theorem 2 and Theorem 3 to this case.*

(b) We mention that if one assumes that $a(x)$ *decays more quickly than given by* $(A1)$ *then one can obtain existence results for (2) for a larger range of* q*; see Remark 2.*

(c) Here we consider the assumption on $a(x)$ *in Theorem 2 part 4. We suspect the linear Liouville theorem assumed in Theorem 2 part 4 should hold assuming only* $(A1)$*, but we are unable to prove this. Here we show that with enough decay assumptions on* $a(x)$ *one does have the required Liouville theorem. Without loss of generality we can assume* $\beta = 1$ *and suppose* ψ *is a smooth solution of* $\Delta\psi(x) - a(x) \cdot \nabla\psi(x) = 0$ *in* $\mathbb{R}^N$.

Claim. *We first claim that a bounded solution* ψ *must be constant and for this we would like to thank Connor Mooney for pointing this out to us. Set* $\phi(x) := \psi(x) - \inf_{\mathbb{R}^N} \psi$ *and note that* ϕ *is a nonnegative bounded solution of the same equation. By the strong maximum principle we can assume that* ϕ *does not attain its supremum or infimum on* $\mathbb{R}^N$. *So we have* $\inf_{\partial B_R} \phi = \inf_{B_R} \phi \to 0$ *as* $R \to \infty$. *For* $\lambda > 0$ *set* $\phi_\lambda(x) := \phi(\lambda x)$ *and so* $0 = \Delta\phi_\lambda(x) - (\lambda a(\lambda x)) \cdot \nabla\phi_\lambda(x)$ *in* $\Omega_0 := \{x \in \mathbb{R}^N : \frac{1}{4} < |x| < 4\}$. *Note that exists some* $C > 0$ *(independent of* λ*) such that* $\lambda|a(\lambda x)| \le C$ *in* Ω_0. *By Harnack's inequality there is some* C *such that*

$$\sup_{\Omega_{00}} \phi_\lambda \le C \inf_{\Omega_{00}} \phi_\lambda,$$

where $\Omega_{00} := \{x \in \mathbb{R}^N : \frac{1}{2} < |x| < 2\}$. *By the maximum principle (and using the above inequality) gives*

$$\sup_{|x|<2} \phi(\lambda x) = \sup_{\Omega_{00}} \phi(\lambda x) \le C \inf_{|x|=2} \phi(\lambda x) \to 0,$$

as $\lambda \to \infty$. *From this we see* $\sup_{B_R} \phi \to 0$ *as* $R \to \infty$ *and hence we must have* $\phi = 0$ *implying* ψ *is constant. This completes the proof of the claim.*

We now show under sufficient decay assumptions on $a(x)$ *that* ψ *is bounded and then from the above we see* ψ *is constant. Recall that* $-\Delta\psi(x) = -a(x) \cdot \nabla\psi(x) =: g(x)$ *(which is smooth) in* $\mathbb{R}^N$ *and recalling we have decay on* $\nabla\psi(x)$ *and assuming decay assumptions on* $a(x)$ *we see that* $g(x)$ *can be made to decay as quickly as we like. Define*

$$\tilde{\psi}(x) := \int_{\mathbb{R}^N} \frac{C_N}{|y|^{N-2}} g(x-y)dy$$

and note that $\tilde{\psi}$ *is bounded (provided* g *decays quick enough). So we have* $\psi - \tilde{\psi}$ *is harmonic in* $\mathbb{R}^N$ *and note that it grows at most sublinearly at* $|x| = \infty$. *From this we can conclude that* $\psi - \tilde{\psi} =$ *constant and hence* $\psi = \tilde{\psi} -$*constant and hence* ψ *is bounded.*

(d) *Here we mention that the linear theory developed in [8, 10] (which closely corresponds to our linear theory on X_2) is enough to handle (1) and (2) in the case of $q < 2$. To consider (2) in the case of $q > 2$ one needs to consider a different class of function spaces and this naturally brings up some interesting Liouville theorems. We believe these new function spaces and associated linear theory is our main contribution in this work.*

We mention that in the works [8, 10, 9] the case of $\frac{N+2}{N-2} < p \le \frac{N+1}{N-3}$ was also examined but one needs extra arguments and for this reason we choose to omit this case. We also mention that many of these ideas extend to exterior domains and this is examined [1]. In that work we were unable to handle (2) in the case of $q \ge 2$.

1.2 Outline of the approach

For the outline of our approach we consider an equation which includes both (1) and (2) as special cases. Consider the equation

$$-\Delta u + a(x) \cdot \nabla u + V(x)u = u^p + \gamma_1 u^{q_1} + \gamma_2 |\nabla u|^{q_2}, \quad \text{in } \mathbb{R}^N. \tag{6}$$

To find a positive classical solution of (6) it is sufficient, via the scaling $v(x) = \lambda^{\frac{-2}{p-1}} u(\lambda^{-1}x) = \lambda^{-\alpha} u(\lambda^{-1}x)$, to find a positive solution of

$$-\Delta v + a^\lambda(x) \cdot \nabla v + V^\lambda(x)v = v^p + \gamma_1 \lambda^{\theta_1} v^{q_1} + \gamma_2 \lambda^{\theta_2} |\nabla v|^{q_2}, \text{ in } \mathbb{R}^N, \tag{7}$$

for some $\lambda > 0$ where $\theta_1 := \frac{2(q_1-p)}{p-1}$ and $\theta_2 := \frac{(p+1)q_2-2p}{p-1}$ and where $a^\lambda(x) := \lambda^{-1} a(\lambda^{-1}x)$ and $V^\lambda(x) := \lambda^{-2} V(\lambda^{-1}x)$. Instead of solving (7) directly we replace the v^p with $|v|^p$ and we will show $v > 0$ after. We look for solutions of the form $v = w + \phi$; hence ϕ will need to satisfy

$$\begin{aligned} -L_\lambda(\phi) &= |w+\phi|^p - w^p - pw^{p-1}\phi \\ &\quad + \gamma_1 \lambda^{\theta_1} |w+\phi|^{q_1} + \gamma_2 \lambda^{\theta_2} |\nabla w + \nabla \phi|^{q_2} \\ &\quad - a^\lambda \cdot \nabla w - V^\lambda w, \end{aligned} \tag{8}$$

where

$$L_\lambda(\phi) = L(\phi) - T_\lambda(\phi) = L(\phi) - a^\lambda(x) \cdot \nabla \phi - V^\lambda(x)\phi.$$

We will look for solutions of (8) in the case of small $\lambda > 0$ and we will treat the terms $\lambda^{\theta_1}|w + \phi|^{q_1}$ and $\lambda^{\theta_2}|\nabla w + \nabla \phi|^{q_2}$ as perturbation terms. This will require that θ_1 and θ_2 are both positive. To solve (8) the approach will be to apply a fixed point argument and hence the invertibility of L_λ, on a suitable space, will be crucial. By Theorem A [8] $L : X_0 \to Y_2$ has a continuous right inverse. Our idea is to view L_λ as a perturbation of L in the Fredholm sense; $L_\lambda = L - T_\lambda$ where T_λ is a compact operator. Of course $T_\lambda : X_0 \to Y_2$ is not a compact operator and so this forces us to adjust the spaces involved. We now define a suitable Banach space. Define $X_2 := \left\{ \phi \in C^1(\mathbb{R}^N \backslash \{0\}) \cap H^2_{loc}(\mathbb{R}^N \backslash \{0\}) : \Delta \phi \in C(\mathbb{R}^N \backslash \{0\}), \text{ and } \|\phi\|_{X_2} < \infty \right\}$, where

$$\begin{aligned} \|\phi\|_{X_2} &:= \sup_{0<|x|\le 1} \left(|x|^\sigma |\phi(x)| + |x|^{\sigma+1} |\nabla \phi(x)| + |x|^{\sigma+2} |\Delta \phi(x)| \right) \\ &\quad + \sup_{|x| \ge 1} \left(|x|^\alpha |\phi(x)| + |x|^{\alpha+1} |\nabla \phi(x)| + |x|^{\alpha+2} |\Delta \phi(x)| \right). \end{aligned}$$

We then show that $T_\lambda : X_2 \to Y_2$ is a compact operator and hence L_λ and L have the same Fredholm index. Using this we are able to show the existence of a continuous right inverse of $L_\lambda : X_2 \to Y_2$ for sufficiently small λ (whose norm is bounded in λ). This result will essentially rely on a Liouville theorem of the form: the only smooth solution of $-\Delta\psi + a \cdot \nabla\psi + V\psi = 0$ in $\mathbb{R}^N$ with $|\psi(x)| \le |x|^{-\sigma}$ is $\psi = 0$. This result will follow directly from the maximum principle. We now return to the specific cases (1) and (2).

In Section 2 we consider (1) using the approach which is outlined above which results in Theorem 1. We comment that the needed linear theory is the right inverse of $L_\lambda : X_2 \to Y_2$ whose operator norm is bounded independently of small λ.

In Section 3 we consider (2). Using a fixed point argument on X_2, as outlined above, we are able to obtain a positive solution for $\frac{2p}{p+1} < q < 2$; which is part 2 of Theorem 2. The condition $\frac{2p}{p+1} < q$ is completely natural since this condition is equivalent to $\theta_2 > 0$ in (6). The restriction $q < 2$ is not expected and is not related to the the equation (2) but rather is a result from our choice of space to perform a fixed point argument. To allow for larger values of q we need to apply a fixed point argument on a space whose functions are less singular near the origin. Picking a space whose norm includes a term like $\sup_{B_1} |\nabla\phi|$ is a good choice for obtaining positive solutions of (2) for any $q > \frac{2p}{p+1}$, provided $a(x) = 0$. This accounts for Theorem 2 part 1 and the function space we use here is denoted by Z_∞, see Lemma 3.
Another choice we consider is a space which allows slightly more blow up at the origin; Z_1 where $Z_\lambda := \{\phi \in C(\mathbb{R}^N) \cap C^1(\mathbb{R}^N\backslash\{0\}) : \|\phi\|_{Z_\lambda} < \infty\}$ where

$$\|\phi\|_{Z_\lambda} := \sup_{|x|\le 1} \left(|\phi(x)| + \lambda^{1-\sigma}|x|^\sigma|\nabla\phi(x)|\right) + \sup_{|x|\ge 1} \left(|x|^\alpha|\phi(x)| + |x|^{\alpha+1}|\nabla\phi(x)|\right).$$

To apply a fixed point argument on Z_1 we need to show that $L_\lambda : Z_1 \to Y_1$ has a continuous right inverse, which is bounded independent of λ, for small $\lambda > 0$, where $Y_1 := \{f \in C(\mathbb{R}^N\backslash\{0\}) : \|f\|_{Y_1} < \infty\}$ and

$$\|f\|_{Y_1} := \sup_{0<|x|\le 1} |x|^{\sigma+1}|f(x)| + \sup_{|x|\ge 1} |x|^{\alpha+2}|f(x)|.$$

We are able to prove this result up to a needed linear Liouville theorem which we assume; this accounts for Theorem 2 part 4.
Without assuming the needed linear Liouville theorem we can weaken slightly the space Z_1 to obtain the needed result. We prove $L_\lambda : Z_\lambda \to Y_1$ has a continuous right inverse whose norm is bounded independently of λ, for small $\lambda > 0$. Note that the space Z_λ gives slightly weaker estimates on the gradient after considering the fact that λ is small. This will be sufficient to apply a fixed point argument in Z_λ and enables us to find positive solutions of (2) provided $q \ge p$. This is given in Theorem 2 part 3.
We now come to Theorem 3. Here the approach is the same is in the proof of Theorem 2 part 3; a fixed point argument in Z_λ. The only difference now is we won't take $\sigma > 0$ small. This allows us to gain a larger range of q. With this larger range of σ one needs to be a bit careful when applying various elliptic regularity results.

2 Equation (1); $-\Delta u + a \cdot \nabla u + Vu = u^p + \gamma u^q$

In this section we obtain positive solutions of (1). The main issue will be to obtain a continuous right inverse for $L_\lambda : X_2 \to Y_2$ whose norm is bounded above by some constant independent for all sufficiently small λ. The first subsection develops this theory and then we move on to the fixed point argument.

2.1 The linear theory of $L_\lambda(\phi) := \Delta\phi + pw^{p-1}\phi - a^\lambda \cdot \nabla\phi - V^\lambda \phi : X_2 \to Y_2$

Before examining L_λ on the desired spaces we need to examine the operator $L(\phi) = \Delta\phi + pw^{p-1}\phi$.

Lemma 1. *Suppose $N \geq 4$ and $p > \frac{N+1}{N-3}$. For $0 < \sigma < N-2$ there exists some $C > 0$ such that for every $f \in Y_2$ there exists some $\phi \in X_2$ such that $L(\phi) = f$ in $\mathbb{R}^N \backslash \{0\}$. In addition $\|\phi\|_{X_2} \leq C \|f\|_{Y_2}$.*

Proof. Let $f \in Y_2$. By Corollary A there is some $\phi \in X_1$ and $C > 0$ (independent of f and ϕ) such that $\|\phi\|_{X_1} \leq C\|f\|_{Y_2}$. Using the equation $L(\phi) = f$ in $\mathbb{R}^N \backslash \{0\}$ directly along with the above X_0 bounds on ϕ and the asymptotic behaviour of w at $r = \infty$ gives

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2} |\Delta\phi(x)| + \sup_{|x|\geq 1} |x|^{\alpha+2} |\Delta\phi(x)| \leq C_2 \|f\|_{Y_2}.$$

Combining this with the X_1 bounds on ϕ, from Corollary A, gives the desired result. □

It follows from Lemma 1 and the fact that $L : X_2 \to Y_2$ is continuous that L has a continuous right inverse $F : Y_2 \to X_2$. Consequently, $\tilde{X}_2 := F(Y_2)$ is a closed subspace of X_2, hence a Banach space with the norm of X_2. In fact we can decompose X_2 as $X_2 = ker(L) \oplus \tilde{X}_2$ where $ker(L)$ is the finite dimensional kernel of L. In particular notice that $L : \tilde{X}_2 \to Y_2$ is a operator with Fredholm index zero. We now wish to examine the linear operator $L_\lambda = L - T_\lambda : \tilde{X}_2 \to Y_2$. The first step will be showing the mapping T_λ is compact.

Lemma 2. *For each fixed $0 < \lambda < \infty$ the operator $T_\lambda : X_2 \to Y_2$ is compact.*

Proof. Set $T^i : X_2 \to Y_2$ by $T^1(\phi) := a(x) \cdot \nabla\phi(x)$ and $T^2(\phi) := V(x)\phi(x)$ and note that if T^i is compact for $i = 1, 2$, then so is T_λ for all $\lambda > 0$.
Consider a sequence $\phi_m \in X_2$ with $\|\phi_m\|_{X_2} \leq C$. Then note that we have $|\Delta\phi_m(x)|, |\nabla\phi_m(x)|, \phi_m(x)$ bounded provided we stay away from the origin. So we see that for any $q < \infty$ we have ϕ_m bounded in $W^{2,q}(\delta \leq |x| \leq \delta^{-1})$ for any $\delta > 0$. So by a diagonal argument we see that there is a subsequence $\{\phi_m\}_m$ (which we don't rename) which is convergent in $C^{1,\frac{1}{2}}(\delta \leq |x| \leq \delta^{-1})$ for all $\delta > 0$. We now show that $T^1(\phi_m)$ is Cauchy in Y_2.

Let $\varepsilon > 0$ be small and fix $R > 1$ big enough such that $\sup_{|z|\geq R} |z||a(z)| \leq \varepsilon$. Then we have

$$\begin{aligned}
\sup_{|x|\geq 1} |x|^{2+\alpha}|T^1(\phi_m) - T^1(\phi_k)| &\leq \sup_{1\leq|x|\leq R} |x|^{2+\alpha}|a(x)\cdot\nabla(\phi_m(x) - \phi_k(x))| \\
&\quad + \sup_{R\leq|x|} |x|^{2+\alpha}|a(x)\cdot\nabla(\phi_m(x) - \phi_k(x))| \\
&\leq R^{2+\alpha} \sup_{1\leq|x|\leq R} |a(x)| \sup_{1\leq|x|\leq R} |\nabla(\phi_m(x) - \phi_k(x))| \\
&\quad + \sup_{R\leq|x|} |x||a(x)||x|^{1+\alpha}|\nabla(\phi_m(x) - \phi_k(x))| \\
&\leq R^{2+\alpha} \sup_{1\leq|x|\leq R} |a(x)| \sup_{1\leq|x|\leq R} |\nabla(\phi_m(x) - \phi_k(x))| \\
&\quad +\varepsilon 2C.
\end{aligned}$$

From this we see that

$$\limsup_{k,m\to\infty} \left(\sup_{|x|\geq 1} |x|^{2+\alpha}|T^1(\phi_m) - T^1(\phi_k)| \right) \leq 2\varepsilon C$$

but $\varepsilon > 0$ was arbitrary and hence the limit is zero. We now consider the other portion of the Y_2 norm.
Fix $\varepsilon > 0$ small and let $0 < \delta < 1$ be small such that $\sup_{|x|\leq\delta} |x||a(x)| \leq \varepsilon$. Now note that

$$\begin{aligned}
\sup_{|x|\leq 1} |x|^{2+\sigma}|T^1(\phi_m) - T^1(\phi_k)| &\leq \sup_{|x|\leq\delta} |x||a(x)| \left(|x|^{1+\sigma}|\nabla\phi_m(x) - \nabla\phi_k(x)|\right) \\
&\quad + \sup_{\delta\leq|x|\leq 1} |x|^{2+\sigma}|a(x)||\nabla\phi_m(x) - \nabla\phi_k(x)| \\
&\leq \varepsilon 2C + \sup_{\delta\leq|x|\leq 1} |x|^{2+\sigma}|a(x)| \sup_{\delta\leq|x|\leq 1} |\nabla\phi_m(x) - \nabla\phi_k(x)|
\end{aligned}$$

and again using the convergence of the gradients away from the origin we see that

$$\lim_{k,m\to\infty} \left(\sup_{|x|\leq 1} |x|^{2+\sigma}|T^1(\phi_m) - T^1(\phi_k)| \right) \leq 2\varepsilon C.$$

So combining with the previous result we have $\limsup_{k,m\to\infty} \|T^1(\phi_m) - T^1(\phi_k)\|_{Y_2} = 0$ and hence $\{T^1(\phi_m)\}_m$ is Cauchy in Y_2. An identical argument shows that T^2 is a compact operator. □

We now state our main linear result for this section.

Proposition 1. *($L_\lambda : X_2 \to Y_2$) Let $N \geq 4$, $p > \frac{N+1}{N-3}$ and suppose (A1) and (A2) hold. Then for $0 < \sigma < N-2$ there is some $\lambda_0 > 0$ small such that for all $0 < \lambda < \lambda_0$ the operator $L_\lambda : \tilde{X}_2 \to Y_2$ is continuous, one to one and onto with continuous inverse. In addition the norm of the inverse is bounded above by a constant independent of λ.*

Proof. Fix $0 < \sigma < N-2$. Recall that $L_\lambda = L - T_\lambda : \tilde{X}_2 \to Y_2$, T_λ is compact and hence the Fredholm index of L_λ is equal to the Fredholm index of L, which is zero. So we first suppose

that L_λ is not onto Y_2 for small λ. Hence there is some $\lambda_m \searrow 0$ such that $L_m := L_{\lambda_m}$ is not onto Y_2 and hence there is some $\phi_m \in \tilde{X}_2$ with $\|\phi_m\|_{X_2} = 1$ such that $L_m(\phi_m) = 0$ in $\mathbb{R}^N \backslash \{0\}$. Now suppose that L_λ is onto but the inverse operator is not bounded uniformly in λ. Then there is some $\lambda_m \searrow 0$ and $f_m \in Y_2$, $\phi_m \in \tilde{X}_2$ such that $L_m(\phi_m) = f_m$ in $\mathbb{R}^N \backslash \{0\}$, $\|f_m\|_{Y_2} \to 0$ and $\|\phi_m\|_{X_2} = 1$. Now note we can view the first case as a special case of the second case.
Hence if either condition fails there is some $\lambda_m \searrow 0$, $f_m \to 0$ in Y_2, $\phi_m \in \tilde{X}_2$, $\|\phi_m\|_{X_2} = 1$ such that $L_m(\phi_m) = f_m$ in $\mathbb{R}^N \backslash \{0\}$. We now derive a contradiction.
We re-write $L_m(\phi_m) = f_m$ as $L(\phi_m) = f_m + a^{\lambda_m}(x) \cdot \nabla \phi_m + V^{\lambda_m}(x)\phi_m$. Using the linear theory for L we see that there is some $C > 0$ such that

$$C\|\phi_m\|_{X_2} \leq \|f_m\|_Y + \|a^{\lambda_m} \cdot \nabla \phi_m + V^{\lambda_m}\phi_m\|_{Y_2}. \tag{9}$$

We now examine in detail the second Y_2 norm on the right hand side. A computation shows that

$$\|a^{\lambda_m} \cdot \nabla \phi_m + V^{\lambda_m}\phi_m\|_{Y_2} = I_m + J_m,$$

where I_m is the portion of the norm in the unit ball and J_m is the portion outside the unit ball. We first estimate J_m,

$$\begin{aligned}
J_m &\leq \sup_{|x|\geq 1} |x|^{2+\alpha} \left(|a^{\lambda_m}(x)||\nabla \phi_m| + V^{\lambda_m}(x)|\phi_m(x)| \right) \\
&\leq \sup_{|x|\geq 1} \frac{|x|}{\lambda_m} |a(\frac{x}{\lambda_m})||x|^{\alpha+1}|\nabla \phi_m| \\
&\quad + \sup_{|x|\geq 1} \frac{|x|^2}{\lambda_m^2} V(\frac{x}{\lambda_m})|x|^\alpha |\phi_m| \\
&\leq (A(\lambda_m^{-1}) + \tilde{V}(\lambda_m^{-1}))\|\phi_m\|_{X_2}.
\end{aligned}$$

Let $0 < \varepsilon_0$ be small which we pick later and we now write $I_m \leq I_m^1 + I_m^2 + I_m^3$ where

$$\begin{aligned}
I_m^1 &:= \sup_{|x|\leq \lambda_m \varepsilon_0} |x|^{\sigma+2} \left(|a^{\lambda_m}(x)||\nabla \phi_m(x)| + V^{\lambda_m}(x)|\phi_m(x)| \right) \\
&\leq \sup_{|z|\leq \varepsilon_0} \left(|z||a(z)| + |z|^2|V(z)| \right) \|\phi_m\|_{X_2}.
\end{aligned}$$

We now define

$$I_m^2 := \sup_{\lambda_m \varepsilon_0 \leq |x| \leq \lambda_m \varepsilon_0^{-1}} |x|^{\sigma+2} \left(|a^{\lambda_m}(x)||\nabla \phi_m(x)| + V^{\lambda_m}(x)|\phi_m(x)| \right),$$

which we will later show goes to zero via a Liouville theorem. Finally we define

$$I_m^3 := \sup_{\varepsilon_0^{-1}\lambda_m \leq |x| \leq 1} |x|^{\sigma+2} \left(|a^{\lambda_m}(x)||\nabla \phi_m(x)| + V^{\lambda_m}(x)|\phi_m(x)| \right).$$

We estimate I_m^3 exactly as we did in the case of J_m to see that

$$I_m^3 \leq \left(A(\varepsilon_0^{-1}) + \tilde{V}(\varepsilon_0^{-1}) \right) \|\phi_m\|_{X_2}.$$

Now fix $0 < \varepsilon_0$ small enough such that

$$\sup_{|z|\le\varepsilon_0} \left(|z||a(z)| + |z|^2|V(z)|\right) + A(\varepsilon_0^{-1}) + V(\varepsilon_0^{-1}) \le \frac{C}{2}.$$

We can then combine the above estimates to arrive at

$$\frac{C}{2}\|\phi_m\|_{X_2} \le \|f_m\|_{Y_2} + \left(A(\lambda_m^{-1}) + \tilde{V}(\lambda_m^{-1})\right)\|\phi_m\|_{X_2} + I_m^2,$$

which gives us a contradiction if we can show that $I_m^2 \to 0$.
We now define the rescaled functions $\psi_m(x) := \lambda_m^\sigma \phi_m(\lambda_m x)$. Let $\varepsilon_k \searrow 0$ with $\varepsilon_0 > \varepsilon_1$ and set $A_k := \{x \in \mathbb{R}^N : \varepsilon_k < |x| < \varepsilon_k^{-1}\}$. Using the bound $\sup_{|x|\le1}|x|^\sigma|\phi_m(x)| \le 1$ one sees that $|\psi_m(x)| \le |x|^{-\sigma}$ on $|x| < \lambda_m^{-1}$. In particular, for each $k \ge 0$ we have ψ_m bounded on A_{k+1} for large enough m. Also note that ψ_m satisfies

$$\Delta\psi_m(x) - a(x)\cdot\nabla\psi_m(x) - V(x)\psi_m(x) = \lambda_m^{\sigma+2}f_m(\lambda_m x) - pw(\lambda_m x)^{p-1}\lambda_m^2\psi_m(x) \quad \text{in } A_{k+1}. \tag{10}$$

Define $g_m(x)$ to be the right hand side of (10) and note that $g_m \to 0$ uniformly in A_{k+1}. Fix $t > N$ large. By elliptic regularity theory there is some $C_{t,k}$ such that

$$\|\psi_m\|_{W^{2,t}(A_k)} \le C_{t,k}\|g_m\|_{L^t(A_{k+1})} + C_{t,k}\|\psi_m\|_{L^1(A_{k+1})}.$$

Note that the right hand side of this inequality is bounded by some $\tilde{C}_k$. So using a diagonal argument and the Sobolev imbedding we can assume that $\{\psi_m\}_m$ (after passing to a suitable subsequence) is bounded in $C^{1,\frac{3}{4}}(A_k)$ for each $k \ge 0$ and there is some $\psi : \mathbb{R}^N\backslash\{0\} \to \mathbb{R}$ such that $\psi_m \to \psi$ in $C^{1,\frac{1}{2}}_{loc}(\mathbb{R}^N\backslash\{0\})$. This is enough to pass to the limit in (10) to see that ψ is a weak solution of

$$\Delta\psi - a(x)\cdot\nabla\psi - V(x)\psi(x) = 0 \text{ in } \mathbb{R}^N\backslash\{0\}.$$

Also note that $\psi(x)$ satisfies the pointwise bounds $|\psi(x)| \le |x|^{-\sigma}$ and $|\nabla\psi(x)| \le |x|^{-\sigma-1}$. Since $0 < \sigma < N-2$ we can apply Lemma 7 to see that ψ is a distributional solution of $\Delta\psi - a(x)\cdot\nabla\psi - V(x)\psi = 0$ in $\mathbb{R}^N$. We can then apply distributional elliptic regularity theory to see that ψ is smooth on $\mathbb{R}^N$. We now show $\psi = 0$. Firstly note that since ψ is smooth and decays to zero as $|x| \to \infty$, we can apply the strong maximum principle to see that $\psi \ge 0$ in $\mathbb{R}^N$. We can then apply the maximum principle to see that $\sup_{B_R}\psi = \sup_{\partial B_R}\psi \to 0$ as $R \to \infty$. This shows that $\psi = 0$. We now recall that $\psi_m \to \psi = 0$ in $C^{1,\frac{1}{2}}(A_k)$ for any k and in particular for $k = 0$. Note that we can estimate I_m^2 as

$$\begin{aligned} I_m^2 &:= \sup_{\lambda_m\varepsilon_0 \le |x| \le \lambda_m\varepsilon_0^{-1}} |x|^{\sigma+2}\left(|a^{\lambda_m}(x)||\nabla\phi_m(x)| + V^{\lambda_m}(x)|\phi_m(x)|\right), \\ &= \sup_{\varepsilon_0\le|z|\le\varepsilon_0^{-1}} |z|^{\sigma+2}\left(|a(z)||\nabla\psi_m(z)| + V(z)|\psi_m(z)|\right) \to 0, \end{aligned}$$

as $m \to \infty$ after considering the above mentioned convergence of ψ_m. This shows that $I_m^2 \to 0$ which gives us the desired contradiction.

□

2.2 Equation (1); the fixed point argument

To find a positive classical solution of (1) it is sufficient (via a scaling argument) to find a positive classical solution v of

$$-\Delta v + a^\lambda(x) \cdot \nabla v + V^\lambda(x) v = v^p + \gamma \lambda^\theta v^q \qquad \mathbb{R}^N, \tag{11}$$

for some $\lambda > 0$ where $\theta := \frac{2(q-p)}{p-1} > 0$. To do this we will find a positive classical solution of

$$-\Delta v + a^\lambda(x) \cdot \nabla v + V^\lambda(x) v = |v|^p + \gamma \lambda^\theta |v|^q \qquad \text{in } \mathbb{R}^N. \tag{12}$$

Considering our function spaces X_2 and Y_2 are spaces defined on the punctured domain we first solve

$$-\Delta v + a^\lambda(x) \cdot \nabla v + V^\lambda(x) v = |v|^p + \gamma \lambda^\theta |v|^q \qquad \text{in } \mathbb{R}^N \backslash \{0\}. \tag{13}$$

To do this we look for solutions of the form $v = w + \phi$ where w is as in the previous sections. Then we need ϕ to satisfy

$$\begin{aligned} L_\lambda(\phi) &= a^\lambda(x) \cdot \nabla w + V^\lambda(x) w \\ &\quad - \left(|w+\phi|^p - w^p - p w^{p-1} \phi \right) \\ &\quad - \gamma \lambda^\theta |w + \phi|^q \qquad \text{in } \mathbb{R}^N \backslash \{0\}. \end{aligned}$$

To solve this equation for ϕ we apply a fixed point argument on a suitable closed ball in $\tilde{X}_2$ centered at the origin. Fix $0 < \sigma$ small and let $0 < \lambda_0$ be as promised by Proposition 1. Given $0 < \lambda < \lambda_0$ and $\phi \in \tilde{X}_2$ define $J_\lambda(\phi) =: \psi_\lambda \in \tilde{X}_2$ where ψ_λ satisfies

$$\begin{aligned} L_\lambda(\psi_\lambda) &= a^\lambda(x) \cdot \nabla w + V^\lambda(x) w \\ &\quad - \left(|w+\phi|^p - w^p - p w^{p-1} \phi \right) \\ &\quad - \gamma \lambda^\theta |w + \phi|^q \qquad \text{in } \mathbb{R}^N \backslash \{0\}. \end{aligned} \tag{14}$$

We will now show that J_λ is a contraction on the closed ball of radius R centered at the origin in $\tilde{X}_2$, which we denote by B_R, for suitable $0 < R$ and $0 < \lambda < \lambda_0$.

Into. Let $\phi \in B_R$. We now estimate the terms on the right hand side of (14). By Lemma 6 we have

$$\left| |w+\phi|^p - p w^{p-1} \phi - w^p \right| \le C \left(w^{p-2} \phi^2 + |\phi|^p \right).$$

Set $\Gamma = |w+\phi|^p - p w^{p-1} \phi - w^p$. Then one sees

$$\begin{aligned} \|\Gamma\|_{Y_2} &\le C \sup_{|x| \le 1} |x|^{\sigma+2} \left(w^{p-2} \phi^2 + |\phi|^p \right) \\ &\quad + C \sup_{|x| \ge 1} |x|^{\alpha+2} \left(w^{p-2} \phi^2 + |\phi|^p \right) \\ &:= C I_1 + C I_2. \end{aligned}$$

$$\begin{aligned} I_1 &= \sup_{|x| \le 1} \left(|x|^{2-\sigma} w^{p-2} \left(|x|^\sigma \phi(x) \right)^2 + |x|^{\sigma+2-\sigma p} \left(|\phi(x)| |x|^\sigma \right)^p \right) \\ &\le \sup_{|x| \le 1} \left(|x|^{2-\sigma} w^{p-2} \|\phi\|_{X_0}^2 + |x|^{\sigma+2-\sigma p} \|\phi\|_{X_0}^p \right) \\ &\le C \|\phi\|_{X_0}^2 + C \|\phi\|_{X_0}^p \end{aligned}$$

for small enough $\sigma > 0$. One can similarly show, using $2+\alpha = p\alpha$ and $2-\alpha = \alpha(p-2)$, that

$$\begin{aligned} I_2 &\leq \sup_{|x|\geq 1} (|x|^\alpha w)^{p-2} \|\phi\|_{X_0}^2 + \|\phi\|_{X_0}^p \\ &\leq C\|\phi\|_{X_0}^2 + \|\phi\|_{X_0}^p. \end{aligned}$$

For $0 < \sigma$ sufficiently small and since $q \geq p$ a computation shows that $\|w^q\|_{Y_2} \leq C$ and $\||\phi|^q\|_{Y_2} \leq 2\|\phi\|_{X_0}^q$. Hence we have $\||w+\phi|^q\|_{Y_2} \leq C + C\|\phi\|_{X_0}^q$.
We now examine the $\|a^\lambda \cdot \nabla w + V^\lambda w\|_{Y_2}$ term. We decompose $|x| \leq 1$ into $|x| \leq \delta$ and $\delta \leq |x| \leq 1$ where we will specify δ later. Set $M_\lambda(x) := |a^\lambda(x)||\nabla w| + V^\lambda(x) w$. First note that a computation shows

$$\sup_{|x|\geq 1} |x|^{\alpha+2} M_\lambda(x) \leq (A(\lambda^{-1}) + \tilde{V}(\lambda^{-1}))\|w\|_{X_2}.$$

A similar computation shows that

$$\sup_{|x|\leq \delta} |x|^{2+\sigma} M_\lambda(x) \leq C\delta^\sigma,$$

where C depends on $\sup_{z\in\mathbb{R}^N}(|z||a(z)| + |z|^2 V(z))$. A similar computation shows that

$$\sup_{\delta\leq|x|\leq 1} |x|^{2+\sigma} M_\lambda(x) \leq CA(\lambda^{-1}\delta) + C\tilde{V}(\lambda^{-1}\delta)$$

where C is independent of λ and δ. Combining these three estimates gives

$$\|a^\lambda \cdot \nabla w + V^\lambda w\|_{Y_2} \leq CA(\lambda^{-1}\delta) + C\tilde{V}(\lambda^{-1}\delta) + C\delta^\sigma. \tag{15}$$

Combining these results and using the inverse bound on L_λ given in Proposition 1 gives

$$\|\psi_\lambda\|_{X_2} \leq C\delta^\sigma + CA(\lambda^{-1}\delta) + C\tilde{V}(\lambda^{-1}\delta) + C(\|\phi\|_{X_0}^2 + \|\phi\|_{X_0}^p) + C|\gamma|\lambda^\theta(C + \|\phi\|_{X_0}^q).$$

Now using the fact that $\phi \in B_R$ gives

$$\|\psi_\lambda\|_{X_2} \leq C\delta^\sigma + CA(\lambda^{-1}\delta) + C\tilde{V}(\lambda^{-1}\delta) + CR^2 + CR^p + C\lambda^\theta(1 + R^q).$$

So for $J_\lambda(B_R) \subset B_R$ it is sufficient that $0 < \delta < 1$, $0 < \lambda < \lambda_0$ and $0 < R$ satisfy

$$C\delta^\sigma + CA(\lambda^{-1}\delta) + \tilde{V}(\lambda^{-1}\delta) + CR^2 + CR^p + C\lambda^\theta + C\lambda^\theta R^q \leq R. \tag{16}$$

Contraction. Let $\hat{\phi}, \phi \in B_R \subset \tilde{X}_2$ and set $J_\lambda(\hat{\phi}) = \hat{\psi}_\lambda, J_\lambda(\phi) = \psi_\lambda$. First note that by Lemma 6 we have

$$\begin{aligned} |L_\lambda(\hat{\psi}_\lambda - \psi_\lambda)| &\leq C\left(w^{p-2}(|\phi| + |\hat{\phi}|) + |\phi|^{p-1} + |\hat{\phi}|^{p-1}\right)|\hat{\phi} - \phi| \\ &\quad + C\lambda^\theta\left(|w|^{q-1} + |\hat{\phi}|^{q-1} + |\phi|^{q-1}\right)|\hat{\phi} - \phi| =: H_1 + H_2. \end{aligned}$$

We now estimate the Y_2 norms of the right hand side. So we have

$$\sup_{|x|\le 1} |x|^{\sigma+2}|H_1| \le \sup_{|x|\le 1} \left(|x|^2 w^{p-2}|\phi| + |x|^2|\phi|^{p-1} \right) \sup_{|x|\le 1} |x|^\sigma |\hat{\phi}-\phi|$$

where we have dropped the $\hat{\phi}$ terms for simplicity of notation. First note that $\sup_{|x|\le 1} |x|^2 w^{p-2}|\phi| \le R$ and $\sup_{|x|\le 1} |x|^2|\phi|^{p-1} \le R^{p-1}$ since $\sigma > 0$ is small and $0 < w \le 1$. Similarly we have

$$\sup_{|x|\ge 1} |x|^{\alpha+2}|H_1| \le \sup_{|x|\ge 1} \left(|x|^2 w^{p-2}|\phi| + |x|^2|\phi|^{p-1} \right) \sup_{|x|\ge 1} |x|^\alpha |\hat{\phi}-\phi|$$

and a computation shows $\sup_{|x|\ge 1} |x|^2 w^{p-2}|\phi| \le CR$ after considering the asymptotic decay of w. Similarly $\sup_{|x|\ge 1} |x|^2|\phi|^{p-1} \le R^{p-1}$. Combining the above two estimates gives $\|H_1\|_{Y_2} \le C(R+R^{p-1})\|\hat{\phi}-\phi\|_{X_2}$. Similarly one can show that $\|H_2\|_{Y_2} \le C\lambda^\theta \left(1+R^{q-1}\right) \|\hat{\phi}-\phi\|_{X_2}$ for $q \ge p$. Using these estimates and the estimates on L_λ gives that

$$\|\hat{\psi}_\lambda - \psi_\lambda\|_{X_2} \le C\left(R + R^{p-1} + \lambda^\theta(1+R^{q-1}) \right) \|\hat{\phi}-\phi\|_{X_2}.$$

So for J_λ to have a Lipschitz constant on B_R at most $\frac{1}{2}$ it is sufficient that

$$C(R+R^{p-1}) + C\lambda^\theta(1+R^{q-1}) \le \frac{1}{2}. \tag{17}$$

We now pick the parameters R, δ, λ so that J_λ is a contraction on B_R; ie. it is sufficient that (16) and (17) to be satisfied. The approach will be to fix $R > 0$ small and then to fix $0 < \delta < 1$ sufficiently small and then to finally pick $\lambda > 0$ small. Doing this one easily sees they can satisfy both (16) and (17).

Hence we can apply Banach's contraction mapping principle to see that there is some $\phi \in B_R$ such that $J_\lambda(\phi) = \phi$. From this we can conclude that $v = w + \phi$ satisfies (13). Moreover notice that by taking $0 < R < 1$ small enough we can assume that the $v > 0$ in say $\{x : |x| \ge \frac{1}{2}\}$. By taking $\sigma > 0$ smaller, if necessary, we can now apply Lemma 7 to see that v satisfies (12) in the sense of distributions. By taking $\sigma > 0$ smaller again we can now apply elliptic regularity theory to see that v is a classical solution of (12). To complete the proof we need to show that v is positive. The approach depends on the sign of γ.

Case 1. $0 \le \gamma$. One can now apply the strong maximum principle on $B_{\frac{1}{2}}$ to see that $v > 0$ in $B_{\frac{1}{2}}$ and hence $v > 0$ in $\mathbb{R}^N$. This complete the proof of Theorem 1 part 1.

Case 2. $\gamma < 0$. We now fix $0 < R < 1$ as above. Then note that we are free to take λ as small as we like. So let $\lambda_m \searrow 0$ and let $\phi_m \in B_R \subset \tilde{X}_2$ be a fixed point for J_{λ_m} on B_R. We set $v_m := w + \phi_m$ and so v_m is positive outside of $B_{\frac{1}{2}} \subset \mathbb{R}^N$ and satisfies

$$-\Delta v_m + a^{\lambda_m} \cdot \nabla v_m + V^{\lambda_m} v_m = |v_m|^p + \gamma \lambda_m^\theta |v_m|^q \qquad \mathbb{R}^N. \tag{18}$$

Our goal is to show that for large enough m that $v_m \ge 0$ in $\mathbb{R}^N$. Towards this define $\Omega_m := \{x \in \mathbb{R}^N : v_m(x) < 0\}$ and lets assume Ω_m is non-empty for all m. Also note that Ω_m is contained in the unit ball in $\mathbb{R}^N$. Now note that v_m satisfies

$$\begin{cases} -\Delta v_m + a^{\lambda_m} \cdot \nabla v_m + C_m(x) v_m &= |v_m|^p \qquad \text{in } \Omega_m \\ v_m &= 0 \qquad \text{on } \partial\Omega_m, \\ v_m &< 0 \qquad \text{in } \Omega_m, \end{cases} \tag{19}$$

where $C_m(x) := V^{\lambda_m} - \lambda_m^\theta |\gamma| |v_m|^{q-1}$.
We now apply Corollary 1 to see that $v_m \ge 0$ in Ω_m provided

$$\| (div(a^{\lambda_m}) - 2V^{\lambda_m} + 2\lambda_m^\theta |\gamma| |v_m|^{q-1})_+ \|_{L^{\frac{N}{2}}(\Omega_m)} < 2S_N,$$

and hence it is sufficient that

$$\| (div(a^{\lambda_m}) - 2V^{\lambda_m})_+ \|_{L^{\frac{N}{2}}(\Omega_m)} + 2|\gamma| \lambda_m^\theta \| |v_m|^{q-1} \|_{L^{\frac{N}{2}}(\Omega_m)} < 2S_N.$$

Now note that a change of variables shows that it is sufficient that

$$\| (div(a) - 2V)_+ \|_{L^{\frac{N}{2}}(\mathbb{R}^N)} + 2|\gamma| \lambda_m^\theta \| |v_m|^{q-1} \|_{L^{\frac{N}{2}}(\Omega_m)} < 2S_N.$$

Now recalling that v_m is bounded in X_2 we see that for sufficiently small $\sigma > 0$ that the second term converges to zero and hence to have $v_m \ge 0$ in Ω_m for large m it is sufficient that $\| (div(a) - 2V)_+ \|_{L^{\frac{N}{2}}(\mathbb{R}^N)} < 2S_N$. We can now apply the strong maximum principle to (18) see that $v_m > 0$ in $\mathbb{R}^N$. We can then apply elliptic regularity to see that v_m is smooth. We can now remove the absolute values and the proof of Theorem 1 part 2 is complete. □

3 Equation (2); $-\Delta u + a \cdot \nabla u = u^p + \gamma |\nabla u|^q$

In this section we are interested in obtaining positive solutions of the nonlinear problem (2). As before we first perform a scaling argument to show it is sufficient to find a positive solution v of

$$-\Delta v(x) + a^\lambda(x) \cdot \nabla v(x) = v(x)^p + \gamma \lambda^\theta |\nabla v(x)|^q \tag{20}$$

where $\theta = (\frac{2}{p-1} + 1)q - \frac{2}{p-1} - 2$, (in this section θ will always refer to this quantity which differs from the previous section). Instead of solving this we consider

$$-\Delta v(x) + a^\lambda(x) \cdot \nabla v(x) = |v(x)|^p + \gamma \lambda^\theta |\nabla v(x)|^q. \tag{21}$$

We look for solutions of the form $v = w + \phi$ and so we need ϕ to satisfy

$$\begin{aligned} -L_\lambda(\phi) &= |w + \phi|^p - w^p - p w^{p-1} \phi \\ &\quad -a^\lambda \cdot \nabla w + \gamma \lambda^\theta |\nabla w + \nabla \phi|^q \quad \text{in } \mathbb{R}^N \end{aligned} \tag{22}$$

where $L_\lambda(\phi) := \Delta \phi + p w^{p-1} \phi - a^\lambda(x) \cdot \nabla \phi$.

3.1 The linear theory

Proposition 2. *Suppose $N \ge 4$, $p > \frac{N+1}{N-3}$, $0 < \sigma < 1$ and suppose a is a smooth divergence free vector field which satisfies satisfies (A1).*

1. *Suppose the only smooth solutions ψ of $\Delta \psi(x) - a(x) \cdot \nabla \psi(x) = 0$ in $\mathbb{R}^N$, which satisfies $|\nabla \psi(x)| \le \frac{C_0}{|x|^\sigma}$ (for some $C_0 > 0$), are the constant solutions. Then there is some small $0 < \lambda_0$ and $C > 0$ such that for all $0 < \lambda < \lambda_0$ and every $f \in Y_1$ there is some $\phi_\lambda \in Z_1$ such that $L_\lambda(\phi_\lambda) = f$ in $\mathbb{R}^N$. Moreover $\|\phi_\lambda\|_{Z_1} \le C \|f\|_{Y_1}$.*

2. *There is some small* $0 < \lambda_0 < 1$ *and* $C > 0$ *such that for every* $f \in Y_1$ *there is some* $\phi_\lambda \in Z_\lambda$ *such that* $L_\lambda(\phi_\lambda) = f$ *in* $\mathbb{R}^N$*. Moreover* $\|\phi_\lambda\|_{Z_\lambda} \le C\|f\|_{Y_1}$.

Proof. 1. We begin by proving the desired result on the small class $Y_1 \cap L^\infty$ and we then extend to all of Y_1 later; note that $Y_1 \cap L^\infty$ is not dense in Y_1. So show there is some small $0 < \lambda_0 < 1$ and $C > 0$ such that for all $0 < \lambda < \lambda_0$ and $f \in L^\infty(\mathbb{R}^N) \cap Y_1$ there is some $\phi_\lambda \in Z_1$ such that $L_\lambda(\phi_\lambda) = f$ in $\mathbb{R}^N$ and $\|\phi_\lambda\|_{Z_1} \le C\|f\|_{Y_1}$.
We now prove this result. Let $0 < \lambda_0$, and $C > 0$ be from Proposition 1 and suppose $0 < \lambda < \lambda_0$ and $f \in L^\infty(\mathbb{R}^N) \cap Y_1$. Using 1, Lemma 7 there is some $\phi_\lambda \in Z_1$ such that $L_\lambda(\phi_\lambda) = f$ in $\mathbb{R}^N$. In addition we have the estimate that $\|\phi_\lambda\|_{X_2} \le C\|f\|_{Y_2} \le C\|f\|_{Y_1}$. By elliptic regularity we also see there is some C_λ such that $\|\phi_\lambda\|_{Z_1} \le C_\lambda\|f\|_{Y_1}$. We would like to now show this constant C_λ can be taken independently of λ. So towards a contradiction we suppose there is some $f_m \in Y_1 \cap L^\infty(\mathbb{R}^N)$ such that $f_m \to 0$ in Y_1 and there is some $\phi_m \in Z_1 \cap C^{1,\delta}_{loc}(\mathbb{R}^N)$ such that $L_{\lambda_m}(\phi_m) = f_m$ in $\mathbb{R}^N$ with $\lambda_m \to 0$ and $\|\phi_m\|_{Z_1} = 1$. Also note that Proposition 1 we have that $\|\phi_m\|_{X_2} \to 0$ and hence we must have $\sup_{|x|\ge 1}(|x|^\alpha|\phi_m(x)| + |x|^{\alpha+1}|\nabla\phi_m(x)|) \to 0$ and this would then imply $\sup_{|x|\le 1}(|\phi_m(x)| + |x|^\sigma|\nabla\phi_m(x)|) \to 1$. We now try and estimate these two terms to obtain a contradiction. First we fix $\frac{N}{2} < t < \frac{N}{\sigma+1}$ and then notice that $\|f_m\|_{L^t(B_1)} \le C\|f_m\|_{Y_1}$.

$L^\infty(B_1)$ estimate on ϕ_m.
Decompose ϕ_m on B_1 as $\phi_m = \overline{\phi}_m + \hat{\phi}_m$ where

$$\Delta\overline{\phi}_m - a^{\lambda_m} \cdot \nabla\overline{\phi}_m = f_m - pw^{p-1}\phi_m \quad \text{in } B_1, \qquad \overline{\phi}_m = 0 \quad \text{on } \partial B_1, \tag{23}$$

and

$$\Delta\hat{\phi}_m - a^{\lambda_m} \cdot \nabla\hat{\phi}_m = 0 \quad \text{in } B_1, \qquad \hat{\phi}_m = \phi_m \quad \text{on } \partial B_1. \tag{24}$$

We will now use Lemma 1.3 from [2] which we restate for the reader convenience. Suppose Ω a bounded domain in $\mathbb{R}^N$, and $b(x)$ a divergence free vector field defined on Ω and $T > \frac{N}{2}$. Then there is some $C = C(\Omega, N, T)$ (independent of b) such that for all $g \in L^T(\Omega)$, there is a solution w of $-\Delta w + b(x) \cdot \nabla w = g$ in Ω with $w = 0$ on $\partial\Omega$. Moreover $\|w\|_{L^\infty} \le C\|g\|_{L^T}$.
Now we return to our problem. Since a is divergence free we see there is some $C_t > 0$ such that $\sup_{B_1}|\overline{\phi}_m| \le C_t\|f_m - pw^{p-1}\phi_m\|_{L^t(B_1)} \le C_t\|f_m\|_{L^t(B_1)} + C_t p\|\phi_m\|_{L^t(B_1)}$. Using the X_0 estimate on ϕ_m and the prior estimate relating the L^t norm of f_m to the Y_1 norm, gives $\sup_{B_1}|\overline{\phi}_m| \le C_t\|f_m\|_{Y_1} + C_t\|f_m\|_{Y_2}$. Using the maximum principle we see that $\sup_{B_1}|\hat{\phi}_m| \le \sup_{|x|=1}|\phi_m| \le C\|f_m\|_{Y_2}$. Combining the estimates gives $\sup_{|x|\le 1}|\phi_m| \le C\|f_m\|_{Y_1} \to 0$.

Gradient estimates on ϕ_m.
Considering the above $L^\infty(B_1)$ and the earlier X_2 estimate on ϕ_m one sees there must be some $0 < |x_m| \to 0$ such that $|x_m|^\sigma|\nabla\phi_m(x_m)| \ge \frac{3}{4}$. We now consider the rescaled functions $\psi_m(x) := |x_m|^{\sigma-1}(\phi_m(|x_m|x) - \phi_m(|x_m|x_0))$ where $|x_0| = 1$ is fixed. Then note that $\psi_m(x_0) = 0$ and $|\nabla\psi_m(x)| \le |x|^{-\sigma}$ for $|x| \le \frac{1}{|x_m|}$ and $|\nabla\psi_m(z_m)| \ge \frac{3}{4}$ where $z_m := \frac{x_m}{|x_m|} \in S^{N-1}$. Now consider the annuli $A_k := \{x \in \mathbb{R}^N : \frac{1}{k} < |x| < k\}$ where $k \ge 2$ an integer. Using the Arzelá-Ascoli Theorem along with a diagonal argument one can find some $\psi \in C(\mathbb{R}^N\backslash\{0\})$ and a subsequence of ψ_m such that $\psi_m \to \psi$ uniformly on each A_k for $k \ge 2$. In addition

note that ψ_m satisfies

$$\Delta\psi_m(x) - \beta_m a(\beta_m x)\cdot\nabla\psi_m(x) = g_m(x),$$

where

$$g_m(x) := |x_m|^{\sigma+1} f_m(|x_m|x) - pw(|x_m|x)^{p-1}|x_m|^2\psi_m(x) - pw(|x_m|x)^{p-1}|x_m|^{\sigma+1}\phi_m(|x_m|x_0),$$

and $\beta_m := \frac{|x_m|}{\lambda_m}$. Note that $g_m \to 0$ uniformly in A_k for any $k \geq 2$. Using elliptic regularity and the Sobolev imbedding theorem we see that ψ_m is bounded in $C^{1,\delta}(A_k)$ for each $k \geq 2$ and hence we can pass to another subsequence to obtain that $\psi_m \to \psi$ in $C^{1,\delta}(A_k)$ for all $k \geq 2$. One should note that advection term does not cause any problems on the annuli, even if $\beta_m \to \infty$, after considering the assumptions on a. By passing to a subsequence we can assume that $\beta_m \to \beta \in [0,\infty]$.
Hence ψ satisfies the bound $|\nabla\psi(x)| \leq |x|^{-\sigma}$ and $\Delta\psi(x) - \beta a(\beta x)\cdot\nabla\psi(x) = 0$ in $\mathbb{R}^N\backslash\{0\}$ where we interpret this equation as just $\Delta\psi = 0$ in the case of $\beta = \infty$. The bound on ψ near the origin is sufficient to show that ψ is a smooth solution of $\Delta\psi(x) - \beta a(\beta x)\cdot\nabla\psi(x) = 0$ in $\mathbb{R}^N$. We now separate the two cases:
(i) $\beta \in \{0,\infty\}$, (ii) $0 < \beta < \infty$.
In the first case we have $\Delta\psi = 0$ in $\mathbb{R}^N$ with the stated decay assumption on $\nabla\psi$. Hence ψ_{x_i} is a harmonic function on $\mathbb{R}^N$ which decays to zero and hence the maximum principle shows that $\psi_{x_i} = 0$ and hence ψ is constant.
(ii) Set $\phi_0(x) := \psi(\frac{x}{\beta})$ and note $\Delta\phi(x) - a(x)\cdot\nabla\phi(x) = 0$ in $\mathbb{R}^N$ with the desired decay of the gradient. We can then apply the hypothesis to see ϕ is constant and hence ψ is constant. But we now recall that we have $\psi_m \to \psi = 0$ in $C^{1,\delta}(A_K)$ for all $k \geq 2$. In particular we have $|\nabla\psi_m(z_m)| \to 0$ giving us the desired contradiction. This completes the proof of the apriori estimate.
We now extend the result to the full space Y_1. Let σ, C, λ_0 be as above and let $0 < \lambda < \lambda_0$ and $f \in Y_1$. Then there is some $\phi \in X_2$ such that $L_\lambda(\phi) = f$ in $\mathbb{R}^N\backslash\{0\}$ and we can extend the pde to the full space after noting the regularity of f. Also we have $\|\phi\|_{X_2} \leq C\|f\|_{Y_2} \leq C\|f\|_{Y_1}$ from our earlier estimates. Define the continuous cut off of $f(x)$ by $f_m(x)$ where $f_m(x)$ is defined by $f_m(x) = f(x)$ for $|f(x)| \leq m$ and $f(x) = m$ for $f(x) \geq m$ and lastly define $f_m(x) = -m$ for $f(x) \leq -m$. Note that f_m is bounded and for large m, f_m and f may only differ near the origin. In addition note that $\|f_m\|_{Y_1} \leq \|f\|_{Y_1}$. By the above estimates there is some $\phi_m \in Z_1$ such that $L_\lambda(\phi_m) = f_m$ and $\|\phi_m\|_{Z_1} \leq C\|f_m\|_{Y_1} \leq C\|f\|_{Y_1}$. Also note that $f_m \to f$ in Y_2 and hence $\phi_m \to \phi$ in X_2 and hence for all $\varepsilon > 0$ we have $|\nabla\phi_m(x)| \to |\nabla\phi(x)|$ uniformly on $\varepsilon \leq |x| \leq 1$. Using this and the estimates on ϕ_m one sees that $\sup_{|x|\leq 1}|x|^\sigma|\nabla\phi(x)| \leq C\|f\|_{Y_1}$. Also note that since $\phi_m \to \phi$ in X_2 and $\sup_{B_1}|\phi_m| \leq C\|f\|_{Y_1}$ we see that $\sup_{B_1}|\phi| \leq C\|f\|_{Y_1}$. Using this and the X_2 bound on ϕ we see that $\phi \in Z_1$ and $\|\phi\|_{Z_1} \leq C\|f\|_{Y_1}$.

We now give a claim which we will need for the proof of part 2 of this proposition.
Claim. Let $N \geq 4$, $0 < \sigma < 1$ and $0 < \varepsilon \leq 1$. Then there is some $C_1 = C_1(N,\sigma) > 0$ (independent of ε) such that for all $f \in Y_1^{in}$ there is some $\phi \in Z_1^{in}$ such that

$$-\Delta\phi = f \text{ in } B_\varepsilon, \quad \phi = 0 \text{ on } \partial B_\varepsilon. \tag{25}$$

In addition one has the estimate $\|\phi\|_{Z_1^{in}} \leq C_1\|f\|_{Y_1^{in}}$ where $\|\phi\|_{Z_1^{in}} := \sup_{|x|\leq\varepsilon} |x|^\sigma|\nabla\phi(x)|$ and $\|f\|_{Y_1^{in}} := \sup_{|x|\leq\varepsilon} |x|^{\sigma+1}|f(x)|$.
To prove this claim one first sees, via a scaling argument, that it is sufficient to prove the result for $\varepsilon = 1$. A local regularity argument then shows that the only possible problem with the gradient estimate is near the origin. One then applies the same rescaling argument as in the proof of part 1 of the proposition. The main difference now being that since $a(x) = 0$ we have the needed Liouville theorem to complete the proof.

2. We now prove part 2. Let $0 < \sigma < 1$ and let $C, \lambda_0 > 0$ be as promised from Proposition 1 and by taking λ_0 small we can assume it is less than one. Let $f \in Y_1$. By Proposition 1 there is some $\phi_\lambda \in X_2$ such that $L_\lambda(\phi_\lambda) = f$ in $\mathbb{R}^N\backslash\{0\}$. In addition we have that $\|\phi_\lambda\|_{X_2} \leq C\|f\|_{Y_2} \leq C\|f\|_{Y_1}$ and noting that the norms agree outside B_1 we need only obtain estimates inside the unit ball. Additionally note that one can show that $L_\lambda(\phi_\lambda) = f$ in $\mathbb{R}^N$ and not just on the punctured domain. As in the first part of this proposition one can show there is some C such that $\sup_{B_1} |\phi_\lambda| \leq C\|f\|_{Y_1}$. So to complete the proof we need only to obtain the desired gradient bounds on the unit ball.

Gradient estimates on ϕ_λ. First note that using the X_2 bound on ϕ_λ gives $\lambda^{1-\sigma}\sup_{\frac{1}{4}\leq|x|\leq 1} |x|^\sigma|\nabla\phi_\lambda(x)| \leq C\|f\|_{Y_1}$. So we need only prove the gradient estimate in $B_{\frac{1}{4}}$.
The proof will involve splitting $B_{\frac{1}{4}}$ into two regions:
(i) $\lambda\varepsilon < |x| < \frac{1}{4}$, and (ii) $|x| \leq \lambda\varepsilon$,
where $\varepsilon > 0$ will be some small but fixed parameter.

Region (i); $\lambda\varepsilon < |x| < \frac{1}{4}$.
Fix $\varepsilon > 0$ small, $\lambda\varepsilon < |x| < \frac{1}{4}$ and consider the rescaled functions $\psi_\lambda(y) := \phi_\lambda(x + |x|y)$ for $y \in B_{\frac{1}{2}}$. Then note that ψ_λ satisfies

$$\Delta\psi_\lambda(y) - \frac{|x|}{\lambda}a\left(\frac{x+|x|y}{\lambda}\right)\cdot\nabla\psi_\lambda(y) = |x|^2f(x+|x|y) - p|x|^2w(x+|x|y)^{p-1}\psi_\lambda(y) := g_\lambda(y) \qquad \text{in } B_{\frac{1}{2}}. \tag{26}$$

Also note that the gradient term satisfies $\frac{|x|}{\lambda}|a(\frac{x+|x|y}{\lambda})| \leq 2A(\frac{|x|}{2\lambda}) \leq 2\sup_{0\leq R} 2A(R)$. Fix $N < t$. Using elliptic regularity shows that

$$\sup_{B_{\frac{1}{4}}} |\nabla\psi_\lambda| \leq C\|g_\lambda\|_{L^t(B_{\frac{1}{2}})} + C\|\psi\|_{L^1(B_{\frac{1}{2}})}.$$

Writing this estimate out gives

$$\sup_{B_{\frac{1}{4}}} |\nabla\psi_\lambda| \leq C|x|^{1-\sigma}\|f\|_{Y_1} + C\|\psi_\lambda\|_{L^1(B_1)} \leq C|x|^{1-\sigma}\|f\|_{Y_1} + C\|f\|_{Y_1} \leq C\|f\|_{Y_1}.$$

Writing this out in terms of ϕ_λ gives $|x||\nabla\phi_\lambda(x)| \leq C\|f\|_{Y_1}$. Now recall that $\lambda\varepsilon < |x| < \frac{1}{4}$ and hence $\varepsilon^{1-\sigma}\lambda^{1-\sigma}|x|^\sigma|\nabla\phi_\lambda(x)| \leq C\|f\|_{Y_1}$ and so

$$\lambda^{1-\sigma}\sup_{\lambda\varepsilon<|x|<\frac{1}{4}} |x|^\sigma|\nabla\phi_\lambda(x)| \leq \frac{C\|f\|_{Y_1}}{\varepsilon^{1-\sigma}}.$$

Region (ii), $|x| \leq \lambda\varepsilon$.
We now fix $\varepsilon > 0$ sufficiently small such that $1 - C_1 \sup_{B_{2\varepsilon}} |y||a(y)| > 0$; this constant C_1 is from the Claim stated at the end of the proof of part 1 of this proposition. Now consider the rescaling $\psi_\lambda(y) = \phi_\lambda(\lambda y)$ for $|y| < 2\varepsilon$. We decompose ψ_λ on $B_{2\varepsilon}$ via $\psi_\lambda = \psi_\lambda^1 + \psi_\lambda^2$ where

$$\Delta\psi_\lambda^1(y) - a(y) \cdot \nabla\psi_\lambda^1(y) = \lambda^2 f(\lambda y) - p\lambda^2 w(\lambda y)^{p-1}\psi_\lambda \quad \text{in } B_{2\varepsilon}$$

with $\psi_\lambda^1 = 0$ on $\partial B_{2\varepsilon}$ and where $\psi_\lambda^2(y)$ satisfies

$$\Delta\psi_\lambda^2(y) - a(y) \cdot \nabla\psi_\lambda^2(y) = 0 \quad \text{in } B_{2\varepsilon}$$

with $\psi_\lambda^2 = \psi_\lambda$ on $\partial B_{2\varepsilon}$.

Estimate on ψ_λ^1. Re-write the equation for ψ_λ^1 as

$$\Delta\psi_\lambda^1(y) = a(y) \cdot \nabla\psi_\lambda^1(y) + \lambda^2 f(\lambda y) - p\lambda^2 w(\lambda y)^{p-1}\psi_\lambda =: g_\lambda(y) \quad \text{in } B_{2\varepsilon}$$

with $\psi_\lambda^1 = 0$ on $\partial B_{2\varepsilon}$. We now apply the above Claim to see we have

$$\sup_{|y|\leq 2\varepsilon} |y|^\sigma |\nabla\psi_\lambda^1(y)| \leq C_1 \sup_{|y|\leq 2\varepsilon} |y|^{\sigma+1}|g_\lambda(y)|.$$

To estimate the right hand side of this consider

$$\begin{aligned}
\sup_{|y|\leq 2\varepsilon} |y|^{\sigma+1}|g_\lambda(y)| \;&\leq\; \sup_{|y|\leq 2\varepsilon} |y||a(y)||y|^\sigma|\nabla\psi_\lambda^1(y)| \\
&\quad + \sup_{|y|\leq 2\varepsilon} \lambda^2 |y|^{\sigma+1}|f(\lambda y)| \\
&\quad + p\lambda^2 \sup_{|y|\leq 2\varepsilon} |y|^{\sigma+1}|\psi_\lambda(y)| \\
&\leq\; \left(\sup_{B_{2\varepsilon}} |y||a(y)|\right) \sup_{B_{2\varepsilon}} |y|^\sigma|\nabla\psi_\lambda^1|(y)| \\
&\quad + \lambda^{1-\sigma} \sup_{|x|\leq 2\varepsilon} |x|^{\sigma+1}|f(x)| \\
&\quad + \lambda^{2-\sigma} C\|f\|_{Y_2}.
\end{aligned}$$

So combining the above estimates give

$$\left(1 - C_1 \sup_{B_{2\varepsilon}} |y||a(y)|\right) \sup_{|y|\leq 2\varepsilon} |y|^\sigma|\nabla\psi_\lambda^1(y)| \leq C_1\lambda^{1-\sigma} \sup_{|x|\leq 2\varepsilon} |x|^{\sigma+1}|f(x)| + CC_1\lambda^{2-\sigma}\|f\|_{Y_2}, \quad (27)$$

and hence there is some $C_2 = C_2(\varepsilon)$ such that

$$\sup_{|y|\leq 2\varepsilon} |y|^\sigma|\nabla\psi_\lambda^1(y)| \leq C_2\lambda^{1-\sigma}\|f\|_{Y_1}. \quad (28)$$

Estimate on ψ_λ^2. Using the fact that $a(x)$ is a smooth vector field we can apply elliptic regularity theory to see for $t > N$ there is some $C_{t,\varepsilon}$ such that

$$\|\psi_\lambda^2\|_{W^{2,t}(B_\varepsilon)} \leq C_{t,\varepsilon} \int_{B_{2\varepsilon}} |\psi_\lambda^2(y)| dy.$$

The Sobolev imbedding theorem and the maximum principle, along with our uniform estimate on ϕ_λ gives

$$\sup_{B_\varepsilon} |\nabla \psi_\lambda^2(y)| \le C_{t,E} \int_{B_{2\varepsilon}} |\psi_\lambda^2(y)| dy \le \tilde{C}_{\varepsilon,t} \sup_{|y|=2\varepsilon} |\psi_\lambda(y)| = \tilde{C}_{\varepsilon,t} \sup_{|x|=2\lambda\varepsilon} |\phi_\lambda(x)| \le C_{t,\varepsilon} \|f\|_{Y_1}. \tag{29}$$

Combining the ψ_λ^i estimates.
Combining (28) and (29) gives

$$|\nabla \psi_\lambda(y)| \le C\|f\|_{Y_1} \left(1 + \lambda^{1-\sigma}|y|^{-\sigma}\right),$$

for all $0 < |y| < \varepsilon$. Writing this out in terms of ϕ_λ gives

$$\lambda^{1-\sigma} \sup_{|x|<\lambda\varepsilon} |x|^\sigma |\nabla \phi_\lambda(x)| \le (\varepsilon^\sigma + \lambda^{1-\sigma})\|f\|_{Y_1}.$$

This is the desired estimate on region (ii). We can now combine with the estimate on region (i) to see we have

$$\lambda^{1-\sigma} \sup_{B_{\frac{1}{4}}} |x|^\sigma |\nabla \phi_\lambda(x)| \le C\|f\|_{Y_1}.$$

□

We include one last linear result which we will use to prove Theorem 2 part 1; the case of $a(x) = 0$. We mention we really do not need to prove this separately since this result can be proven using another of our approaches, but it is much easier to prove it this way. This will require that we define another class of function spaces. Define

$$\|f\|_{Y_\infty} := \sup_{|x|\le 1} |f(x)| + \sup_{|x|\ge 1} |x|^{\alpha+2}|f(x)|, \quad \text{and}$$

$$\|\phi\|_{Z_\infty} := \sup_{|x|\le 1} (|\phi(x)| + |\nabla\phi(x)|) + \sup_{|x|\ge 1} (|x|^\alpha|\phi(x)| + |x|^{\alpha+1}|\nabla\phi(x)|),$$

and note that our notation for the function spaces Z_λ for $0 < \lambda < \lambda_0$ and Z_1 are consistent with each other but that Z_∞ is not consistent with the others.

Lemma 3. *Let $N \ge 4$ and $p > \frac{N+1}{N-3}$. Then there is some $C > 0$ such that for every $f \in Y_\infty$ there is some $\phi \in Z_\infty$ such that $L(\phi) = f$ in $\mathbb{R}^N$ and $\|\phi\|_{Z_\infty} \le C\|f\|_{Y_\infty}$.*

Proof. The existence of a solution follows from Theorem A and then one applies elliptic regularity to complete the proof. □

3.2 Equation (2); the fixed point arguments

In this section we prove Theorem 2 which contains four different parts. To prove this result we apply fixed point arguments in a variety of different spaces to obtain a solution ϕ of (22). We now formally define the nonlinear mapping. Given ϕ in a suitable space we define $J_\lambda(\phi) =: \psi_\lambda$ where ψ_λ satisfies

$$\begin{aligned} -L_\lambda(\psi_\lambda) &= |w+\phi|^p - w^p - pw^{p-1}\phi \\ &\quad -a^\lambda \cdot \nabla w + \gamma\lambda^\theta|\nabla w + \nabla\phi|^q, \end{aligned} \tag{30}$$

where the domain where the equation is satisfied on is either all of $\mathbb{R}^N$ or the punctured space $\mathbb{R}^N\backslash\{0\}$; which will depend on which function space we are working. To show the J_λ is well defined we require that the right hand side of (30) is in a suitable space and we can then apply the appropriate linear theory; these computations are included below in the part we label *into*. When showing J_λ is contraction mapping there will be, as usual, two portions. One portion will be showing the mapping is into add the other part will be the contraction part. Since we are considering a number of spaces we will collect all the calculations associated with the into portion into one section and the contraction portion into another. After this we return and examine each space individually and perform the contraction mapping on each space. Note that in the case of $a(x) = 0$ that L_λ is just L. To avoid unnecessary duplication we will continue to write this as L_λ. Keep in mind that the spaces Y_∞ and Z_∞ will only be utilized in the case of $a(x) = 0$.

3.2.1 Some computations.

Here we collect various computations that we will need when performing the fixed point arguments later.

Into. For $k \in \{1, 2, \infty\}$ we have

$$\begin{aligned} \|L_\lambda(\psi_\lambda)\|_{Y_k} &\le \| |w+\phi|^p - w^p - pw^{p-1}\phi\|_{Y_k} \\ &\quad + \|a^\lambda \cdot \nabla w\|_{Y_k} + C\lambda^\theta \left(\| |\nabla w|^q \|_{Y_k} + \| |\nabla \phi|^q \|_{Y_k} \right). \end{aligned} \tag{31}$$

We now examine these terms in the various spaces.

(i) $\| |w+\phi|^p - w^p - pw^{p-1}\phi\|_{Y_k}$. In the previous section we have shown that $\| |w+\phi|^p - w^p - pw^{p-1}\phi\|_{Y_2} \le C(\|\phi\|_{X_0}^2 + \|\phi\|_{X_0}^p)$. The identical calculation shows that

$$\| |w+\phi|^p - w^p - pw^{p-1}\phi\|_{Y_1} \le C(\|\phi\|_{Z_\lambda}^2 + \|\phi\|_{Z_\lambda}^p),$$

for $0 < \lambda < \lambda_0$ and $\lambda = 1$. One also sees that

$$\| |w+\phi|^p - w^p - pw^{p-1}\phi\|_{Y_\infty} \le C(\|\phi\|_{Z_\infty}^2 + \|\phi\|_{Z_\infty}^p).$$

(ii) $\|a^\lambda \cdot \nabla w\|_{Y_k}$. Let $0 < \delta < 1$. In the previous section we obtained the estimate $\|a^\lambda \cdot \nabla w + V^\lambda w\|_{Y_2} \le CA(\lambda^{-1}\delta) + CV(\lambda^{-1}\delta) + C\delta^\sigma$. If one carries out the same proof but drops the V^λ term one sees that $\|a^\lambda \cdot \nabla w\|_{Y_2} \le CA(\lambda^{-1}\delta) + C\delta^{\sigma+2}$ where C is independent of δ and $0 < \lambda < \lambda_0$. The identical calculation shows that $\|a^\lambda \cdot \nabla w\|_{Y_1} \le CA(\lambda^{-1}\delta) + C\delta^{\sigma+1}$. As mentioned above we will only utilize Y_∞, Z_∞ in the case of $a(x) = 0$.

(iii) $\| |\nabla w|^q \|_{Y_k} + \| |\nabla \phi|^q \|_{Y_k}$. Here we show some details since this term was not examined in the previous sections. We begin with w. Note

$$\sup_{|x|\ge 1} |x|^{2+\alpha} |\nabla w(x)|^q \le \sup_{|x|\ge 1} |x|^{2+\alpha} \frac{C}{|x|^{(\alpha+1)q}},$$

which is finite provided $q \ge \frac{\alpha+2}{\alpha+1}$. We now consider $|x| \le 1$. Note that since w is smooth we have $\sup_{|x|\le 1} |x|^\beta |\nabla w(x)|^q \le C$ for all $0 \le \beta < \infty$. In particular we have $\| |\nabla w|^q \|_{Y_k} \le C$ for

$k \in \{1, 2, \infty\}$ provided the above condition on q is satisfied. We now examine $\||\nabla\phi|^q\|_{Y_k}$. First note that for $|x| \geq 1$ we have $|x|^{\alpha+1}|\nabla\phi(x)| \leq \|\phi\|_X$ for $X \in \{X_2, Z_1, Z_\lambda, Z_\infty\}$. So we have $\sup_{|x|\geq 1} |x|^{\alpha+2}|\nabla\phi(x)|^q \leq \|\phi\|_X^q$ for $X \in \{X_2, Z_1, Z_\lambda, Z_\infty\}$ provided $q \geq \frac{\alpha+2}{\alpha+1}$. We now consider $|x| \leq 1$ and now the results will depend on which space we are in. First note that for $|x| \leq 1$ we have

$$|x|^{\sigma+1}|\nabla\phi(x)| \leq \|\phi\|_{X_2}, \quad |\nabla\phi(x)| \leq \|\phi\|_{Z_\infty}, \quad \lambda^{1-\sigma}|x|^\sigma|\nabla\phi(x)| \leq \|\phi\|_{Z_\lambda},$$

where the last result holds for $0 < \lambda < \lambda_0$ and $\lambda = 1$. Using these estimates gives

1. for $\frac{\alpha+2}{\alpha+1} \leq q \leq \frac{\sigma+2}{\sigma+1}$ one has $\||\nabla\phi|^q\|_{Y_2} \leq 2\|\phi\|_{X_2}^q$,
2. for $\frac{\alpha+2}{\alpha+1} \leq q \leq \frac{1}{\sigma} + 1$ one has $\||\nabla\phi|^q\|_{Y_1} \leq \|\phi\|_{Z_\lambda}^q \left(\lambda^{-(1-\sigma)q} + 1\right)$ for $0 < \lambda < \lambda_0$ and $\lambda = 1$.
3. for $\frac{\alpha+2}{\alpha+1} \leq q$ one has $\||\nabla\phi|^q\|_{Y_\infty} \leq 2\|\phi\|_{Z_\infty}^q$.

Contraction. Let $\hat{\phi}, \phi \in B_R$ where B_R is the closed ball of radius R centered at the origin in either X_2, Z_λ, Z_1 or Z_∞. We set $\hat{\psi}_\lambda := J_\lambda(\hat{\phi})$ and $\psi_\lambda := J_\lambda(\phi)$. Then

$$\begin{aligned} -L_\lambda(\hat{\psi}_\lambda - \psi_\lambda) &= |w + \hat{\phi}|^p - |w + \phi|^p - pw^{p-1}(\hat{\phi} - \phi) \\ &\quad + \gamma\lambda^\theta(|\nabla w + \nabla\hat{\phi}|^q - |\nabla w + \nabla\phi|^q). \end{aligned} \tag{32}$$

Hence, for $k \in \{1, 2, \infty\}$ we have

$$\begin{aligned} \|L_\lambda(\hat{\psi}_\lambda - \psi_\lambda)\|_{Y_k} &\leq \||w + \hat{\phi}|^p - |w + \phi|^p - pw^{p-1}(\hat{\phi} - \phi)\|_{Y_k} \\ &\quad + |\gamma|\lambda^\theta\||\nabla w + \nabla\hat{\phi}|^q - |\nabla w + \nabla\phi|^q\|_{Y_k}. \end{aligned} \tag{33}$$

We now estimate the various quantities from (33). We begin with the first term on the right, $\||w + \hat{\phi}|^p - |w + \phi|^p - pw^{p-1}(\hat{\phi} - \phi)\|_{Y_k}$.

4. For $\hat{\phi}, \phi \in B_R \subset X_2$ the previous section shows $\||w+\hat{\phi}|^p - |w+\phi|^p - pw^{p-1}(\hat{\phi}-\phi)\|_{Y_2} \leq C(R + R^{p-1})\|\hat{\phi} - \phi\|_{X_2}$.
5. For $\hat{\phi}, \phi \in B_R \subset Z_\lambda$ where $0 < \lambda < \lambda_0$, or $\lambda = 1$ we have, by a similar calculation, $\||w + \hat{\phi}|^p - |w + \phi|^p - pw^{p-1}(\hat{\phi} - \phi)\|_{Y_1} \leq C(R + R^{p-1})\|\hat{\phi} - \phi\|_{Z_\lambda}$.
6. For $\hat{\phi}, \phi \in B_R \subset Z_\infty$ we have $\||w+\hat{\phi}|^p - |w+\phi|^p - pw^{p-1}(\hat{\phi}-\phi)\|_{Y_\infty} \leq C(R+R^{p-1})\|\hat{\phi}-\phi\|_{Z_\infty}$.

We now estimate the nonlinear gradient term; $\||\nabla w + \nabla\hat{\phi}|^q - |\nabla w + \nabla\phi|^q\|_{Y_k}$. Firstly note that by Lemma 6 we have

$$\left||\nabla w + \nabla\hat{\phi}|^q - |\nabla w + \nabla\phi|^q\right| \leq C\left(|\nabla w|^{q-1} + |\nabla\hat{\phi}|^{q-1} + |\nabla\phi|^{q-1}\right)|\nabla\hat{\phi} - \nabla\phi|. \tag{34}$$

From this we see that

$$\sup_{|x|\geq 1} |x|^{\alpha+2}\left||\nabla w + \nabla\hat{\phi}|^q - |\nabla w + \nabla\phi|^q\right| \leq CK_0 \sup_{|x|\geq 1} |x|^{\alpha+1}|\nabla\hat{\phi} - \nabla\phi|, \tag{35}$$

where

$$
\begin{aligned}
K_0 &= \sup_{|x|\geq 1} |x| \left(|\nabla w|^{q-1} + |\nabla \hat{\phi}|^{q-1} + |\nabla \phi|^{q-1} \right) \\
&\leq \sup_{|x|\geq 1} \left((|x|^{\alpha+1}|\nabla w|)^{q-1} + (|x|^{\alpha+1}|\nabla \hat{\phi}|)^{q-1} + (|x|^{\alpha+1}|\nabla \phi|)^{q-1} \right) \\
&\leq C + \sup_{|x|\geq 1} \left((|x|^{\alpha+1}|\nabla \hat{\phi}|)^{q-1} + (|x|^{\alpha+1}|\nabla \phi|)^{q-1} \right)
\end{aligned}
$$

provided $(\alpha+1)(q-1) \geq 1$, which is equivalent to $q \geq \frac{\alpha+2}{\alpha+1}$.

7. Let $\hat{\phi}, \phi \in B_R$ in Z_∞. Then using (34) one sees that $\sup_{|x|\leq 1} \left||\nabla w + \nabla \hat{\phi}|^q - |\nabla w + \nabla \phi|^q\right| \leq C(1+R^{q-1})\|\hat{\phi} - \phi\|_{Z_\infty}$ and combining this with the above estimate for $|x| \geq 1$ we see

$$
\||\nabla w + \nabla \hat{\phi}|^q - |\nabla w + \nabla \phi|^q\|_{Y_\infty} \leq C(1+R^{q-1})\|\hat{\phi} - \phi\|_{Z_\infty},
$$

provided $q \geq \frac{\alpha+2}{\alpha+1}$.

8. Let $\hat{\phi}, \phi \in B_R$ in X_2. Then we have

$$
\sup_{|x|\leq 1} |x|^{\sigma+2} \left||\nabla w + \nabla \hat{\phi}|^q - |\nabla w + \nabla \phi|^q\right| \leq C K_2 \sup_{|x|\leq 1} |x|^{\sigma+1} |\nabla \hat{\phi} - \nabla \phi|,
$$

where $K_2 := \sup_{|x|\leq 1} \left(|x||\nabla w|^{q-1} + |x||\nabla \hat{\phi}|^{q-1} + |x||\nabla \phi|^{q-1} \right)$. Note that $|x|^{\sigma+1}|\nabla \phi(x)| \leq R$ for $|x| \leq 1$ and hence $\sup_{|x|\leq 1} |x||\nabla \phi(x)|^{q-1} \leq R^{q-1}$ provided $1 \geq (\sigma+1)(q-1)$, and then note this condition on q is equivalent to $q \leq \frac{\sigma+2}{\sigma+1}$. So combining this with the above results for $|x| \geq 1$ we see that

$$
\||\nabla w + \nabla \hat{\phi}|^q - |\nabla w + \nabla \phi|^q\|_{Y_2} \leq C(1+R^{q-1})\|\hat{\phi} - \phi\|_{X_2},
$$

provided $\frac{\alpha+2}{\alpha+1} \leq q \leq \frac{\sigma+2}{\sigma+1}$.

9. Let $\hat{\phi}, \phi \in B_R$ in Z_λ where $0 < \lambda < \lambda_0$ or $\lambda = 1$. Then we have

$$
\sup_{|x|\leq 1} |x|^{\sigma+1} \left||\nabla w + \nabla \hat{\phi}|^q - |\nabla w + \nabla \phi|^q\right| \leq \frac{C K_1}{\lambda^{1-\sigma}} \sup_{|x|\leq 1} |x|^{\sigma} \lambda^{1-\sigma} |\nabla \hat{\phi} - \nabla \phi|,
$$

where $K_1 := \sup_{|x|\leq 1} \left(|x||\nabla w|^{q-1} + |x||\nabla \hat{\phi}|^{q-1} + |x||\nabla \phi|^{q-1} \right)$. Note that for $|x| \leq 1$ we have $\lambda^{1-\sigma}|x|^{\sigma}|\nabla \phi(x)| \leq R$ and hence $|x||\nabla \phi(x)|^{q-1} \leq \lambda^{-(1-\sigma)(q-1)} R^{q-1}$ provided $1 \geq \sigma(q-1)$. Combining this with the estimates for outside the unit ball gives

$$
\||\nabla w + \nabla \hat{\phi}|^q - |\nabla w + \nabla \phi|^q\|_{Y_1} \leq C \left(1 + R^{q-1} + \frac{1}{\lambda^{1-\sigma}} + \frac{R^{q-1}}{\lambda^{q(1-\sigma)}} \right) \|\hat{\phi} - \phi\|_{Z_\lambda},
$$

for all $0 < \lambda < \lambda_0$ and $\lambda = 1$.

We now perform the fixed point arguments. For clarity we separate by the various spaces. In all cases we let $\hat{\phi}, \phi \in B_R$ and $J_\lambda(\hat{\phi}) = \hat{\psi}_\lambda, J_\lambda(\phi) = \psi_\lambda$ where B_R is in the appropriate space.

Fixed point argument in Z_∞. Completion of proof of Theorem 2 part 1. Recall in this case we are taking $a(x) = 0$. Since $L_\lambda = L : Z_\infty \to Y_\infty$ has a continuous right inverse and after considering (31), (i) and (iii) we have

$$\|\psi_\lambda\|_{Z_\infty} \leq C(R^2 + R^p) + C\lambda^\theta(1 + R^q), \tag{36}$$

provided $q \geq \frac{\alpha+2}{\alpha+1}$. By (33), 6 and 7 we see

$$\|\hat{\psi}_\lambda - \psi_\lambda\|_{Z_\infty} \leq C\left(R + R^{p-1} + \lambda^\theta + \lambda^\theta R^{q-1}\right)\|\hat{\phi} - \phi\|_{Z_\infty}. \tag{37}$$

Also note that $\theta > 0$ exactly when $q > \frac{\alpha+2}{\alpha+1}$. So for J_λ to be a contraction on B_R in Z_∞ it is sufficient that $C(R^2 + R^p) + C\lambda^\theta(1 + R^q) \leq R$ and $C\left(R + R^{p-1} + \lambda^\theta + \lambda^\theta R^{q-1}\right) \leq \frac{1}{2}$. One easily sees that they can satisfy the two conditions by first fixing $R > 0$ sufficiently small and then taking $\lambda > 0$ sufficiently small. One then can apply Banach's fixed point argument to see there is some $\phi \in B_R$ such that $J_\lambda(\phi) = \phi$ and hence ϕ satisfies (22). We then have that $v = w + \phi$ satisfies (21) in $\mathbb{R}^N$ and by taking $R > 0$ small enough we have that $v = w + \phi > 0$ in $\mathbb{R}^N$. We then see that v is a positive classical solution of (20).

Fixed point argument in X_2. Completion of proof of Theorem 2 part 2. Suppose $\frac{\alpha+2}{\alpha+1} < q < 2$ and hence we can take $\sigma > 0$ sufficiently small such that $\frac{\alpha+2}{\alpha+1} < q < \frac{\sigma+2}{\sigma+1}$. Since $L_\lambda : X_2 \to Y_2$ has a continuous right inverse for small λ whose norm is independent of $0 < \lambda < \lambda_0$ and after considering (31), (i), (ii) and (iii) we have

$$\|\psi_\lambda\|_{X_2} \leq C(R^2 + R^p) + CA(\lambda^{-1}\delta) + C\delta^{\sigma+2} + C\lambda^\theta(1 + R^q),$$

and by (33), 4 and 8 we have

$$\|\hat{\psi}_\lambda - \psi_\lambda\|_{X_2} \leq C(R + R^{p-1} + \lambda^\theta + \lambda^\theta R^{q-1})\|\hat{\phi} - \phi\|_{X_2}.$$

So for J_λ to be a contraction on B_R in X_2 it is sufficient that $C(R^2 + R^p) + CA(\lambda^{-1}\delta) + C\delta^{\sigma+2} + C\lambda^\theta(1 + R^q) \leq R$ and $C(R + R^{p-1} + \lambda^\theta + \lambda^\theta R^{q-1}) \leq \frac{1}{2}$. As in the previous case we can satisfy both conditions provided we fix $R > 0$ sufficiently small and then fix $0 < \delta < 1$ sufficiently small and then lastly take $\lambda > 0$ sufficiently small. We can then apply Banach's fixed point argument to see there is some $\phi \in B_R$ such that $J_\lambda(\phi) = \phi$ and hence ϕ satisfies (22). We then have that $v = w + \phi$ satisfies (21) in $\mathbb{R}^N \backslash \{0\}$ and by taking $R > 0$ small enough we have that $v = w + \phi > 0$ for $|x| \geq 1$, for instance. By taking $\sigma > 0$ small enough (21) is satisfied in a suitable weak sense in $\mathbb{R}^N$. Also note that since $\sigma > 0$ is small the term $|v|^p$ will not cause any regularity issues. The only potential problematic term is the nonlinear gradient term $|\nabla v|^q$. Note that we can re-write (21) as $-\Delta v + a^\lambda \cdot \nabla v = |v|^p + \gamma\lambda^\theta b(x) \cdot \nabla v$ where $b(x) := |\nabla v|^{q-2}\nabla v$. Then note by taking $\sigma > 0$ small enough we have $b \in L^Q_{loc}(\mathbb{R}^N)$ for some $Q > N$. We can then apply elliptic regularity to see that $v \in C^{1,\varepsilon}_{loc}$, for some small $\varepsilon > 0$, and we then apply elliptic regularity again, after noting the right hand side of (21) is Hölder continuous, to see that $v \in C^{2,\varepsilon_0}_{loc}$ for some small $\varepsilon_0 > 0$. We then re-write (21) as $-\Delta v + \big(a^\lambda(x) - \gamma\lambda^\theta b(x)\big) \cdot \nabla v = |v|^p$ and we can then apply the maximum principle to see that $v > 0$ in $\mathbb{R}^N$.

Fixed point argument in Z_1. Completion of proof of Theorem 2 part 4. Recall we are assuming sufficient conditions on $a(x)$ such that there is some continuous right inverse of $L_\lambda : Z_1 \to Y_1$ whose norm is independent of $0 < \lambda < \lambda_0$. Using (31), (i), (ii) and (iii) we have

$$\|\psi_\lambda\|_{Z_1} \leq C(R^2 + R^p) + CA(\lambda^{-1}\delta) + C\delta^{\sigma+1} + C\lambda^\theta(1 + R^q)$$

provided we have $\frac{\alpha+2}{\alpha+2} < q < \frac{1}{\sigma} + 1$. By (33), 5 and 9 we see

$$\|\hat{\psi}_\lambda - \psi_\lambda\|_{Z_1} \leq C\left((R + R^{p-1}) + \lambda^\theta(1 + R^{q-1})\right)\|\hat{\phi} - \phi\|_{Z_1}.$$

Note carefully that the only λ's which are present are from the scaling factor in front of the nonlinear gradient term; λ^θ, they are not coming from a Z_λ norm. So for J_λ to be a contraction on B_R in Z_1 it is sufficient that $C(R^2 + R^p) + CA(\lambda^{-1}\delta) + C\delta^{\sigma+1} + C\lambda^\theta(1 + R^q) \leq R$ and $C\left((R + R^{p-1}) + \lambda^\theta(1 + R^{q-1})\right) \leq \frac{1}{2}$. Note these conditions are precisely the conditions which we needed to apply the fixed point argument in X_2. Note the restriction on q is weaker and by taking $\sigma > 0$ small enough we can obtain a fixed point for J_λ for any $q > \frac{\alpha+2}{\alpha+1}$. One can now carry on as in the previous case to show the solution is sufficiently regular and positive. One comment we make is that once one has a suitable weak solution of $-\Delta v + a^\lambda \cdot \nabla v = |v|^p + \gamma\lambda^\theta|\nabla v|^q$ in $\mathbb{R}^N$ then immediately we obtain a $C^{1,\varepsilon}$ solution, for some small $\varepsilon > 0$, after picking $\sigma > 0$ small enough. To see this note that $|\nabla v|^q \leq C|x|^{-\sigma q}$ in B_1 and hence $|\nabla v|^q \in L^Q_{loc}(\mathbb{R}^N)$ for some $Q > N$ after taking $\sigma > 0$ sufficiently small.

Fixed point argument in Z_λ. Completion of proof of Theorem 2 part 3. There is no need to consider $q \leq 2$ since we can already obtain a positive solution in this case without the divergence free assumption on $a(x)$. Recall that for $0 < \sigma < 1$ that $L_\lambda : Z_\lambda \to Y_1$ has a continuous right inverse whose norm is bounded above by a constant independent of $0 < \lambda$ for small λ. We now assume that $\frac{\alpha+2}{\alpha+1} < q$ and by taking $\sigma > 0$ smaller we can assume that $q < \frac{1}{\sigma} + 1$. Considering this and using (31), (i), (ii) and (iii) gives

$$\|\psi_\lambda\|_{Z_\lambda} \leq C(R^2 + R^p) + CA(\lambda^{-1}\delta) + C\delta^{\sigma+1} + C\lambda^\theta(1 + R^q(\lambda^{-(1-\sigma)q} + 1)).$$

By (33), 5 and 9 we see

$$\begin{aligned}\frac{\|\hat{\psi}_\lambda - \psi_\lambda\|_{Z_\lambda}}{\|\hat{\phi} - \phi\|_{Z_\lambda}} &\leq C(R + R^{p-1}) \\ &\quad + C\lambda^\theta\left(1 + R^{q-1} + \frac{1}{\lambda^{1-\sigma}} + \frac{R^{q-1}}{\lambda^{q(1-\sigma)}}\right).\end{aligned}$$

So for J_λ to be a contraction on B_R in Z_λ it is sufficient that

$$C(R^2 + R^p) + CA(\lambda^{-1}\delta) + C\delta^{\sigma+1} + C\lambda^\theta + C\lambda^{\theta-q(1-\sigma)}R^q + C\lambda^\theta R^q \leq R, \tag{38}$$

and

$$C(R+R^{p-1})+C\lambda^{\theta}\left(1+R^{q-1}+\frac{1}{\lambda^{1-\sigma}}+\frac{R^{q-1}}{\lambda^{q(1-\sigma)}}\right)\leq\frac{1}{2}. \tag{39}$$

As in the previous cases we will satisfy these two conditions by first fixing $R>0$ small and then fixing $0<\delta<1$ small and finally taking $\lambda>0$ sufficiently small. Examining (38) and (39) we see this is possible provided:
(a) $\theta>0$, (b) $\theta-q(1-\sigma)\geq 0$, and (c) $\theta-(1-\sigma)>0$.
Note that since $q\geq 2$ and $\sigma>0$ small (b) implies (a) and (c). Writing out (b) gives $q\geq\frac{\alpha+2}{\alpha+\sigma}$. So for this range of q and by taking $R>0$ small we can argue as the proof where we used the fixed point on Z_1 to see there is a positive $C^{2,\varepsilon}$ solution (for some small $\varepsilon>0$) v of (21). Recall in that proof one needed to take $\sigma>0$ sufficiently small to apply some regularity theory. So there is a positive solution of (21) provided $q\geq\frac{\alpha+2}{\alpha}=p$.

Proof of Theorem 3. If one argues exactly as in the proof of Theorem 2 we see that J_λ is a contraction on B_R in Z_λ provided the various parameters satisfy (38) and (39). As in the previous cases the procedure will be to take fix $R>0$ small and then to fix $0<\delta<1$ small and then take $\lambda>0$ small. It is clear this procedure will work provided:
(i) $\theta>0$, (ii) $\theta-q(1-\sigma)\geq 0$, (iii) $\theta-(1-\sigma)>0$.
We now pick the parameters. Recall we are assuming that q satisfies $q>2$ and (5); which is the condition $\alpha q+\frac{q}{q-1}>\alpha+2$. We now pick $\varepsilon>0$ small such that $\alpha q+\frac{q}{q-1}\geq\alpha+2+\varepsilon q$ and we define $\sigma:=\frac{1}{q-1}-\varepsilon$. By picking $\varepsilon>0$ smaller yet again, we have $0<\sigma<1$ after one considers $q>2$. We now show (i)-(iii) are satisfied. Firstly recall that $\theta>0$ is just the condition that $q>\frac{\alpha+2}{\alpha+1}$ which we are assuming. Since $q>1$ we see that (iii) follows from (ii). A computation shows that (ii) is equivalent to $\alpha q+\frac{q}{q-1}\geq\alpha+2+\varepsilon q$. So for these choices of parameters there exists a fixed point, $\phi\in B_R$, of J_λ. By taking $R>0$ small enough we have that $v=w+\phi$ is positive solution of (20), at least on the punctured domain $\mathbb{R}^N\backslash\{0\}$. Note that v has enough regularity near the origin for it to satisfy (20) in the sense of distributions. We now investigate the regularity of v. For this one needs to perform an iteration argument and for this we consider the following simplified model problem: suppose $0<v_0$ satisfies

$$-\Delta v_0=|\nabla v_0|^q+g(x)\qquad\text{in }\Omega,\qquad v_0=0\text{ on }\partial\Omega, \tag{40}$$

where v_0 satisfies (40) in the sense of distributions and is suitably smooth away from the origin which we assume is contained in Ω. Note that if $|\nabla v_0|^q\in L^T(\Omega)$ for some $T>N$ then elliptic regularity shows that $v_0\in W^{2,T}(\Omega)$ and we can then apply the Sobolev imbedding to see that $|\nabla v_0|^q$ is bounded, and one can then easily see that $v\in C^{2,\delta}(\Omega)$ for some small $\delta>0$. We now perform an iteration to show the following:
if $|\nabla v_0|^q\in L^{T_0}(\Omega)$ for some $T_0>\frac{N}{q'}>1$ (q' is the conjugate index of q) then $|\nabla v_0|^q\in L^T(\Omega)$ for some $T>N$.
Suppose $|\nabla v|^q\in L^{T_n}(\Omega)$ for some $T_n>\frac{N}{q'}$. Then by elliptic regularity we have $v_0\in W^{2,T_n}(\Omega)$ and hence $|\nabla v_0|\in W^{1,T_n}(\Omega)$. If $T_n>N$ then by the Sobolev imbedding theorem we have $|\nabla v_0|$ bounded and we are done. We now suppose $T_n\leq N$. If $T_n=N$ then elliptic regularity shows that $v\in W^{2,N}(\Omega)$ and hence $|\nabla v|\in W^{1,N}(\Omega)\subset L^T(\Omega)$ for all $T<\infty$ and we are done. Now suppose $T_n<N$. Then we have $|\nabla v_0|\in L^{\frac{NT_n}{N-T_n}}(\Omega)$ and hence we have $|\nabla v_0|^q\in L^{T_{n+1}}(\Omega)$

where

$$T_{n+1} := \frac{NT_n}{(N-T_n)q}.$$

Now note that $T_{n+1} > T_n$ provided $T_n > \frac{N}{q'}$. So if $T_0 > \frac{N}{q'} > 1$ then we have $T_n < T_{n+1}$ for $0 \le n$, provided we can continue the iteration. It is also clear that after a finite number of iterations there is some $T_n < N$ such that $T_{n+1} \ge N$. If $T_{n+1} > N$ then we are done and the case of equality is covered above.
We now return to the case of (20). First notice that for $\varepsilon > 0$ small we have

$$\frac{N}{q'} < \frac{N}{\frac{q}{q-1} - \varepsilon q}.$$

Fix T_0 to be strictly between these quantities. Then note that since $|\nabla v(x)|^q \le \frac{C}{|x|^{\frac{q}{q-1}-\varepsilon q}}$ for $|x| \le 1$, and so $|\nabla v|^q \in L^{T_0}_{loc}(\mathbb{R}^N)$. We can then apply the above result for the model problem (the proof of the model problem easily extends to the case of (20), to see that $|\nabla v|$ is locally bounded. We can then easily obtain that v is a $C^{2,\delta}$ solution for some small δ.

□

Remark 2. *We now give a rough outline how one can obtain a positive solution of (2) in the case where $a(x)$ has added decay. We argue exactly as in the proof of Theorem 3. To show J_λ has a fixed point in B_R in Z_λ we require that*

$$C(R^2 + R^p) + CA(\lambda^{-1}\delta) + C\delta^{\sigma+1} + C\lambda^\theta + C\lambda^{\theta - q(1-\sigma)}R^q + C\lambda^\theta R^q \le R, \tag{41}$$

and

$$C(R + R^{p-1}) + C\lambda^\theta \left(1 + R^{q-1} + \frac{1}{\lambda^{1-\sigma}} + \frac{R^{q-1}}{\lambda^{(1-\sigma)q}}\right) \le \frac{1}{2}, \tag{42}$$

are satisfied. The difference in the current argument is we now choose the parameters R and δ in a different manner. We choose $R = R(\lambda) = \lambda^t$ and $\delta = \delta(\lambda) = \varepsilon\lambda^{\frac{t}{\sigma+1}}$ where $\varepsilon > 0$ is chosen small and where $t > 0$ is picked later. Then note that with these choices of R and δ we satisfy (41) and (42) provided $\varepsilon > 0$ is sufficiently small and:
1) $\frac{A(\lambda^{-1}\delta(\lambda))}{\lambda^t} \to 0$ *as* $\lambda \searrow 0$,
2) $\theta > t$, *3)* $\theta - q(1-\sigma) + qt > t$, *4)* $\theta + qt > t$, *5)* $\theta + t(q-1) > 0$,
6) $\theta - 1 + \sigma > 0$, *7)* $\theta + t(q-1) - q(1-\sigma) > 0$.
To satisfy 1) we will require that $\lambda^{-1}\delta(\lambda) \to \infty$ as $\lambda \searrow 0$; which requires that $t < \sigma + 1$. Once this is satisfied that we can satisfy 1) by imposing enough decay conditions on $a(x)$; we omit the numerology.

4 Appendix

The following lemma follows from elliptic regularity and Sobolev imbedding. See, for instance, Lemma 2.2 [20].

Lemma 4. *Suppose $A_1 \subset\subset A_2$ are bounded concentric annuli or balls in $\mathbb{R}^N$.*

1. *Suppose $1 < t < \infty$ and ϕ is a distribution defined on A_2 such that the right hand side of (43) is finite. Then $\phi \in W^{2,t}(A_1)$ and one has the estimate*

$$\|\phi\|_{W^{2,t}(A_1)} \leq C \left(\int_{A_2} |\Delta\phi(x)|^t dx \right)^{\frac{1}{t}} + C \int_{A_2} |\phi(x)| dx. \tag{43}$$

   In addition we have C depending only on t, A_1, A_2

2. *For $N < t < \infty$ one has*

$$\sup_{A_1} |\nabla\phi| \leq C \left(\int_{A_2} |\Delta\phi(x)|^t dx \right)^{\frac{1}{t}} + C \int_{A_2} |\phi(x)| dx, \tag{44}$$

   where C depends on t, A_1, A_2.

We now recall the particular maximum principle but this requires we recall the best constant S_N associated with the critical Sobolev imbedding $H_0^1 \subset L^{2^*}$ which is independent of the domain; $S_N \|\phi\|_{L^{2^*}}^2 \leq \|\nabla\phi\|_{L^2}^2$ for all $\phi \in H_0^1$.

Lemma 5. ***Maximum Principle.*** *[15] Suppose $w \in H_0^1(\Omega)$ is a weak solution of $-\Delta w(x) - C(x)w = f(x) \geq 0$ in Ω where $\|C_+\|_{L^{\frac{N}{2}}(\Omega)} < S_N$. Then $w \geq 0$ in Ω.*

For our purposes we need a slight adjustment of this result.

Corollary 1. *Suppose $w \in H_0^1$ satisfies $-\Delta w + a(x) \cdot \nabla w + C(x)w = f \geq 0$ in Ω and*

$$\|(div(a) - 2C)_+\|_{L^{\frac{N}{2}}(\Omega)} < 2S_N.$$

Then $w \geq 0$ in Ω.

Proof. We follow the proof of [15]. Multiply the equation by w_- and integrate by parts to arrive at

$$S_N \int_\Omega w_-^{2^*} dx \leq \int_\Omega |\nabla w_-|^2 dx \leq \int_\Omega \left(\frac{div(a)}{2} - C \right) w_-^2 dx \leq \int_\Omega \left(\frac{div(a)}{2} - C \right)_+ w_-^2 dx,$$

where we used the critical Sobolev imbedding on the left. Now apply Hölder's inequality on the right and combine terms.

□

Lemma 6. *Suppose $p > 1$. There exists a constant $C > 0$ such that the following hold:*

1. *For all numbers $w > 0$, $\phi \in \mathbb{R}$, and $\hat{\phi}$,*

$$\Big| |w + \phi|^p - pw^{p-1}\phi - w^p \Big| \leq C \Big(w^{p-2}\phi^2 + |\phi|^p \Big),$$

   and

$$\Big| |w + \hat{\phi}|^p - |w + \phi|^p - pw^{p-1}(\hat{\phi} - \phi) \Big| \leq C \Big(w^{p-2}(|\phi| + |\hat{\phi}|) + |\phi|^{p-1} + |\hat{\phi}|^{p-1} \Big) |\hat{\phi} - \phi|;$$

2. For all $x, y, z \in \mathbb{R}^n$,

$$\Big| |x+y|^p - |x+z|^p \Big| \le C\Big(|x|^{p-1} + |y|^{p-1} + |z|^{p-1} \Big) |y-z|.$$

We now come to a slight generalization of some well known results regarding extending distributional solutions from a punctured domain to the full space.

Lemma 7. *Suppose $3 \le N$, $0 < \sigma < N-2$, $f \in L^1_{loc}(\mathbb{R}^N), C \in L^\infty_{loc}(\mathbb{R}^N)$, $a \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and $\phi \in L^1_{loc}(\mathbb{R}^N \backslash \{0\})$ satisfies $\Delta \phi + a(x) \nabla \phi + C(x) \phi = f$ in $\mathbb{R}^N \backslash \{0\}$ in the sense of distributions. Suppose exists some $C_0 > 0$ such that $|\phi(x)||x|^\sigma \le C_0$ for all $0 < |x| < 1$. Then $\Delta \phi + a(x) \cdot \nabla \phi + C(x) \phi = f(x)$ in $\mathbb{R}^N$ in the sense of distributions.*

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