1

A ~~glob~~ local solution

Cond A $\quad f : \mathbb{R}^2 \longrightarrow \mathbb{R} \quad \forall R > 0 \quad \exists C_R > 0$

$$|f(t,x) - f(t,y)| \leq C_R |x - y| \quad \forall t$$

$\forall \; |x|, |y| \leq R,$

$$\begin{cases} y'(t) = f(t, y(t)) & 0 \leq t \leq T \\ y(0) = y_0 \end{cases} \qquad (1)$$

$\exists y \in C[0,T]$

$$\begin{cases} y(t) = y_0 + \int_0^t f[\tau, y(\tau)] d\tau & \forall 0 \leq t \leq T \end{cases} \qquad (2)$$

(2)

Fix $0 < R \leq 1$, $0 < T$, $y \in C[0,T] = C(I)$ $\quad I = [0,T]$

$$M(y)(t) := y_0 + \int_0^t f(\tau, y(\tau))\, d\tau.$$

Try showing M contraction

on $B_R(y_0) = \{ y \in C[0,T] : \|y - y_0\|_{C(I)} \leq R \}$

into $y \in B_R(y_0)$

$$M(y)(t) - y_0 = \int_0^t f[\tau, y(\tau)]\, d\tau$$

$$|M(y)(t) - y_0| \leq \int_0^t |f(\tau, y(\tau))|$$

$$|f(t,x)| \leq C_R |x| + |f(t,0)|$$

$\forall |x| \leq R \qquad t \in \mathbb{R}.$

(3)

$$|m(y)(t) - y_0| \le \int_0^t \left[C_R |y(\tau)| + |f(t, 0)| \right] dt$$

~~By taking T small can Assume~~

~~$|y(\tau)| \le 2R$. $|y(\tau) - y$~~

$$|y(t) - y_0| \le R$$

$$|y(t)| \le |y_0| + R$$

$$|m(y)(t) - y_0| \le \int_0^t \left(C_R \left[|y_0| + R \right] + |f(\tau, 0)| \right) d\tau$$

$$\le C_R \left[|y_0| + R \right] T + T \sup_{0 \le t \le T} |f(t, 0)|.$$

(4)

$$\sup_{0\le t\le T} |m(y)(t)-y_0| \le C_R(|y_0|+R)T$$

$$+ T \sup_{0\le t\le T} |f(t,0)|.$$

s.t.f.

$$C_R(|y_0|+R)T + T\sup_{0\le t\le T}|f(t,0)| \le R$$ *

contraction $\bar{y}, y \in B_R(y_0)$

$$m(\bar{y})(t)-m(y)(t) = \int_0^t f(\tau,\bar{y}(\tau))-f(\tau,y(\tau))\,d\tau$$

$$|m(\bar{y})(t)-m(y)(t)| \le \int_0^t |f(\tau,\bar{y}(\tau))-f(\tau,y(\tau))|\,d\tau$$

$$\le C_{|y_0|+R}\int_0^t |\bar{y}(\tau)-y(\tau)|\,d\tau.$$

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$$\| m(y^1) - m(y) \|_{C(I)} \le C_{(|y_0|+r)}\, T\, \| y^1 - y \|_{C(I)}$$

want $\boxed{C_{|y_0|+r}\, T \le \frac{3}{4}}$ **

Maximal solutions (initial value thm)

~~Example~~ 2.1

f satisfies Cond A.

$$\begin{cases} y'(t) = f(t, y(t)) & 0 \le t \\ y(0) = y_0 \end{cases} \qquad (3)$$

Blow up Alternative

One of the following holds:

① $\exists y \in C[0, \infty)$ solves integral formulation of (3).

② $\exists\, 0 < T < \infty$ $\exists\, y \in C[0, T)$ solves integral formulation of (3) & $\lim_{t \to T^-} |y(t)| = \infty$.

example of Blowing up solution

Joining two solutions

let

$$\begin{cases} y'(t) = f[t, y(t)] & 0 \le t \le T \\ y(0) = y_0 \end{cases}$$

$$\begin{cases} \tilde{y}'(t) = f[t, \tilde{y}(t)] & T \le t \le \tilde{T} \\ \tilde{y}(T) = y(T). \end{cases}$$

set $z(t) = \begin{cases} y(t) & 0 \le t \le T \\ \tilde{y}(t) & T \le t \le \tilde{T}. \end{cases}$

then z solve

$$\begin{cases} z'(t) = f(t, z(t)) & 0 \le t \le T \\ z(0) = y_0. \end{cases}$$

idea keep joining the solutions.

NEED some ~~set~~ theoretic ideas to do.

Aside

$\mathcal{C} =$ a family of subintervals on $[0,1)$ s.t. for $\forall x \in [0,1)$ $\exists y \in (x,1)$ s.t. $[x,y) \in \mathcal{C}$.

then $\exists$ countable disjoint subfamily $\mathcal{E} \subset \mathcal{C}$ s.t.

$$\bigcup_{[x,y) \in \mathcal{E}} [x,y) = [0,1).$$

Maximal solution via Zorns Lem.

<u>Zorns lemma</u>

A partially ordered set containing upper bounds for each chain, contains a maximal element.

Recall we are solving

$$\begin{cases} y'(t) = f(t, y(t)) & 0 \le t. \\ y(0) = y_0 \end{cases} \qquad (1)$$

let $\mathcal{C}$ denote the collection of $(y, [0, T))$ that solve (1)

ie $y \in C[0, T)$ & y solves (1)

Given $(y, [0,T)) \;\&\; (\tilde{y}, [0,\tilde{T})) \in \mathcal{C}$

we say

$(y, [0,T)) \underset{p}{\le} (\tilde{y}, [0,\tilde{T}))$ if

partial order

$T \le \tilde{T}$ & $y(t) = \tilde{y}(t)$ $0 \le t < T$.

Then $(\mathcal{C}, \le)$ a partially ordered set.

let $\{(y_\alpha, [0, T_\alpha)) : \alpha \in A\}$

denote a chain (a linearly ordered set,

IE $\forall \alpha, \beta \in A$

we have

$$(y_\alpha, [0, T_\alpha)) \leq (y_\beta, [0, T_\beta))$$

or vice versa.

NEED TO find an upper bound.

~~let~~

$$[0, \tilde{T}) = \bigcup_{\alpha \in A} [0, T_\alpha).$$

For $0 \leq t < \tilde{T}$ fix $\alpha \in A$

s.t. $t < T_\alpha$ & set

$\tilde{Y}(t) = y_\alpha(t)$. Doesn't matter which we choose by uniqueness

Claim 9)

$(\tilde{Y}, [0, \tilde{T}))$ is upper bound for chain.

By Zorns lemma $\exists$ maximal element; ie some $y \in C([0,T))$

which solves ODE & if

$z \in C[0,T_0)$ solves ODE ~~not~~ we have $T_0 \le T$. (so no way to extend past T)

Assume $T < \infty$.

Need to show

$\varlimsup_{t \to T^-} |y(t)| = \infty$; ie $\forall\, t_m \to T$

$|y(t_m)| \longrightarrow \infty$.

$\boxed{10 \text{ pp}}$

Suppose <u>not</u>; then $\exists\ t_n \to T$

$\exists\ C > 0$

$|y(t_n)| \leq C.$

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Now consider the ODE

$$\begin{cases} z'(t) = f[t, z(t)] & t_m \le t \quad (m) \\ z(t_m) = y(t_m) := y_m. \end{cases}$$

By local existence $\exists\, \delta_m > 0$ s.t. we can solve (m) on $[t_m, t_m + \delta_m]$. Idea is to show δ_m bounded from below, hence for m large we have

$t_m + \delta_m > T$.

But then we can joint the two solutions to get a

(12)

solution of

$$\begin{cases} y'(t) = f(t, y(t)) & 0 \le t \le t_m + d_m \\ y(0) = y_0 \end{cases}$$

contradicting maximality of z_0 (need to define partial ordering.)

Go back & look at existence;

T depends on R & C_R

Chapter 3 Second order linear equations.

3.1 Homogeneous second order constant co-efficient ODE's.

$a, b, c \in \mathbb{R}$ $\quad a \neq 0$

$$a y''(t) + b y'(t) + c y(t) = 0. \qquad (1)$$

Look for a solution of the form

$y(t) = e^{rt}$ $\quad$ r to be determined.

$$0 \overset{\text{want}}{=} a y'' + b y' + c y$$

$$= a r^2 e^{rt} + b r e^{rt} + c e^{rt}$$

$$= e^{rt} \left[a r^2 + b r + c \right]$$

So $y(t) = e^{rt}$ solves (1)

iff r satisfies the associated characteristic equation

$ar^2 + br + c = 0.$

Ex $y''(t) + y'(t) - 2y(t) = 0.$ (2)

Find <u>some</u> solutions.

Look for solutions $y(t) = e^{rt}$

$$r^2 + r - 2 = 0$$
$$\| $$
$$(r+2)(r-1)$$

$\Rightarrow r = -2, \; r = 1.$

So $y_1(t) = e^{-2t}$
$y_2(t) = e^{t}$
both solve ODE.

By linearity we see

$y(t) = c_1 y_1(t) + c_2 y_2(t)$ solves

the ODE for any $c_i \in \mathbb{R}$.

To see whether this is <u>ALL</u>

solutions we need to

investigate further.

Linear independence & Wronskian.

let f, g diff. on open interval,

define

$$W(f,g)(t) = \det \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

$$= f(t)g'(t) - f'g.$$

Recall f, g Linear dependent on I

if $\exists\, c_i \in \mathbb{R}$

$c_1 f + c_2 g = 0$ on I

ie $c_1 f(t) + c_2 g(t) = 0$ on I.

(17)

If f, g linear dependent on I $\Longleftrightarrow$ $W(f,g)(t) \equiv 0$ on I.

proof

$(\Longrightarrow)$

$\exists c \in \mathbb{R}$ $g = cf$.

$$W(f,g)(t) = fg' - f'g$$
$$= fcf' - f'cf = 0.$$

$(\Longleftarrow)$ $0 = W(f,g)(t)$

$$fg' = f'g$$

$$\frac{g'}{g} = \frac{f'}{f} \Rightarrow \frac{d}{dt}\log(g) = \frac{d}{dt}\log(f)$$

$$\Rightarrow \log(g) = \log(f) + c$$
$$\Rightarrow \boxed{g = cf}$$

General second order linear Homogeneous

Consider

$$y''(t) + p(t)y'(t) + q\,y(t) = 0. \qquad (1)$$

<u>Claim</u>. If y_1 & y_2 solve (1) & $\{y_1, y_2\}$ is Linearly independent then the general solution of (1) is

$$y(t) = c_1 y_1 + c_2 y_2.$$

<u>Lemma</u>. There is at most one solution of (1) with

$y(0) = y_0 \in \mathbb{R}$ & $y'(0) = y_0' \in \mathbb{R}$.

Proof of lemma Let y_1 & y_2 solve (1) with stated initial conditions.

let $W(t) = W(y_1, y_2)(t)$

$$= \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

$$= y_1 y_2' - y_1' y_2$$

Start Here.

$$W'(t) = y_1' y_2' + y_1 y_2'' - y_1' y_2' - y_1'' y_2$$

$$= y_1 y_2'' - y_1'' y_2$$

use ODE now

$$= y_1(-p y_2') + y_2 p y_1'$$

$$= p\,[-W]$$

So

$$\begin{cases} W'(t) + p(t)W(t) = 0 \\ W(0) = y_1 y_2' - y_1' y_2 \Big|_{t=0} = 0. \end{cases}$$

Then we know $W(t) \equiv 0$ by 1^{st} order ODE theory

$\Rightarrow \{y_2, y_1\}$ is linearly dependent,

so $\exists\, c$ s.t.

$y_2 = c y_1$ but to satisfy initial condition $\Rightarrow c = 1$

$\Rightarrow y_2 = y_1$.

So at most one solution of initial value problem.

[2]

<u>proof of claim</u>

let $\tilde{y}$ solve (1), & let $\{y_1, y_2\}$ be linearly independent solutions of (1).

<u>Goal</u> show $\exists c_i$

$\tilde{y} = c_1 y_1 + c_2 y_2.$

Consider the initial value problem

$$\begin{cases} y'' + p(t) y' + c y = 0 \\ y(0) = \tilde{y}(0) \\ y'(0) = \tilde{y}'(0) \end{cases} \qquad (2)$$

lets find c_1 & c_2 s.t.

$y = c_1 y_1 + c_2 y_2$ solves (2)

but of course $\tilde{y}$ solves (2)

so by uniqueness of IVP

$\Rightarrow \tilde{y} = y = c_1 y_1 + c_2 y_2$.

Only thing to find is $c_i \in \mathbb{R}$

s.t.

$\tilde{y}(0) = c_1 y_1(0) + c_2 y_2(0)$ &

$\tilde{y}'(0) = c_1 y_1'(0) + c_2 y_2'(0)$.

NOTE since $\{y_1, y_2\}$ Linearly independent

$\Rightarrow$ ~~$w(t)$~~ $w(t)$

$w(y_1, y_2)(t) \neq 0$ all t

$\Rightarrow w(y_1, y_2)(0) \neq 0$

$$\Rightarrow \det \begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} \neq 0$$

$$\Rightarrow \exists\, c \in \mathbb{R}$$

$$\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{Y}(0) \\ \vec{Y}'(0) \end{pmatrix}$$

~~Comment If y_1, y_2 satisfy~~

3.4 Complex roots of characteristic eq.

$a \neq 0$

$$ay'' + by' + cy = 0 \qquad (1)$$

$$y(t) = e^{rt}$$

$$ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. $b^2 - 4ac > 0$ (2 real roots)
2. $b^2 - 4ac = 0$ (1 real root)
3. $b^2 - 4ac < 0$ 2 complex roots.

3 $r = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac}$

$= \frac{-b}{2a} \pm \frac{i}{2a}\sqrt{4ac-b^2}$

$\lambda := \frac{-b}{2a}, \quad \mu = \frac{\sqrt{4ac-b^2}}{2a}.$

$r = \lambda \pm i\mu$

$y(t) = e^{(\lambda \pm i\mu)t}$

$= e^{\lambda t} e^{\pm i\mu t}$

$= \cancel{e^{\lambda t}}$

$y(t) = e^{\lambda t} e^{i\mu t}$

$e^{i\mu t} = \cos(\mu t) + i\sin(\mu t)$

$$y_1(t) = e^{\lambda t}\cos(\mu t) + ie^{\lambda t}\sin(\mu t).$$

$$y_2(t) = e^{\lambda t}\, e^{i(-\mu t)}$$

$$= e^{\lambda t}\left[\cos(\mu t) - i \sin(\mu t)\right].$$

$$= e^{\lambda t}\cos(\mu t) - i\, e^{\lambda t}\sin(\mu t).$$

Add them

$$\hat{y}(t) = 2e^{\lambda t}\cos(\mu t).$$

take differen

$$\tilde{y}(t) = 2ie^{\lambda t}\sin(\mu t).$$

set $y_1(t) = e^{\lambda t}\cos(\mu t)$

$y_2(t) = e^{\lambda t}\sin(\mu t).$

Manually check y_i solves (1)

& clearly

$\{y_1, y_2\}$ Linearly independent;

so the general solution of (1)

is $y(t) = c_1 y_1(t) + c_2 y_2(t)$.

3.5 ~~Repeated Roots~~

Ex complex roots case

~~Repeated~~

3.5 Repeated roots case

$ay'' + by' + cy = 0 \qquad y = e^{rt}$

$ar^2 + br + c = 0$

$$r = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a}$$

$y_1(t) = e^{rt}$ a solution

start here

look for second solution of form

$y(t) = y_1(t)\, \vartheta(t)$

ϑ unknown.

$y' = y_1' \vartheta + y_1 \vartheta'$

$y'' = y_1'' \vartheta + 2 y_1' \vartheta' + y_1 \vartheta''$

want

$$0 = ay'' + by' + cy$$

$$= a[\cancel{y_1'' \gamma} + 2y_1'\gamma' + y_1\gamma'']$$

$$+ b(\cancel{y_1'\gamma} + y_1\gamma') + c\,\cancel{y_1\gamma}$$

$$= \gamma\,\underbrace{[ay_1'' + by_1' + cy_1]}_{=0}$$

$$+ a2y_1'\gamma' + ay_1\gamma'' + by_1\gamma'$$

so need

$$0 = (ay_1)\gamma'' + [by_1 + 2ay_1']\gamma'$$

note this is second order

but morally 1st order.

plug in y_1 now

$$0 = ae^{rt}\gamma'' + [be^{rt} + 2are^{rt}]\gamma'$$

$$0 = \nu'' + \left(\frac{b}{a} + 2r\right)\nu'$$

$2r = \frac{-b}{a}$ so

$$0 = \nu'' + \left(\frac{b}{a} - \frac{b}{a}\right)\nu' = \nu''$$

$\Rightarrow$ $\nu(t) = At + B$ so

$$y(t) = \nu y_1 = e^{rt}[At + B]$$

$$= A\,te^{rt} + \underbrace{B\,e^{rt}}_{\text{Already known this solves it.}}$$

$y_2(t) = te^{rt}$ solves ODE &

clearly $\{y_1, y_2\} = \{e^{rt}, te^{rt}\}$

linearly independent.

so $y(t) = c_1 e^{rt} + c_2 t e^{rt}$ the general solution.

<u>Ex</u> (perturbing two real zeros).

$(r-1)^2 = 0$

$r^2 - 2r + 1 = 0$

$y'' - 2y' + y = 0$ $\quad$ $y_1(t) = e^t$

$y_2(t) = te^t$

solve IVP $\left\{ \begin{array}{l} y'' - 2y' + y = 0 \\ y(0) = 0 \\ y'(1) = 0 \end{array} \right.$

$y(t) = te^t$

Try perturbing.

Try $(r-1)(r-(1+\varepsilon))=0$

$r^2 - r(1+\varepsilon) - r + 1+\varepsilon = 0$

$r^2 + (-2-\varepsilon)r + 1+\varepsilon = 0$

$$\begin{cases} y_\varepsilon'' - (2+\varepsilon)y_\varepsilon' + (1+\varepsilon)y_\varepsilon = 0 \\ y_\varepsilon(0) = 0 \quad y_\varepsilon'(0) = 1. \end{cases}$$

$y_\varepsilon(t) = C_\varepsilon e^t + \widehat{C_\varepsilon} e^{(1+\varepsilon)t}$

solve for C_ε & $\widehat{C_\varepsilon}$ to get

~~$y_\varepsilon(t) =$~~ $\widehat{C_\varepsilon} = \frac{1}{\varepsilon}$ $\quad C_\varepsilon = -\frac{1}{\varepsilon}$

then

$y_\varepsilon(t) = \frac{1}{\varepsilon} e^{(1+\varepsilon)t} - \frac{1}{\varepsilon} e^t$

$= te^t \left[\frac{e^{\varepsilon t} - 1}{t\varepsilon} \right] \to te^t \cdot 1.$