So the general sol. of ODE is

Scan

$$y(t) = y_1(t)\int \frac{-y_2(\tau)\, f(\tau)}{W(\tau)}\, d\tau$$

$$+ y_2(t)\int \frac{y_1(\tau)\, f(\tau)}{W(\tau)}\, d\tau$$

$$+ c_1\, y_1(t) + c_2\, y_2(t).$$

Euler equations

$$\cancel{a x^2 y''(x) + b\, y'(x)\, x + c\, y(x) = f(x)}$$

$$a t^2 y''(t) + b\, t\, y'(t) + c\, y(t) = f(t)$$

Homogeneous

$$a t^2 y''(t) + b t y'(t) + c y(t) = 0$$

Try solution of form

$y(t) = t^{\alpha}$ $\alpha \in \mathbb{R}$.

$y' = \alpha t^{\alpha-1}$ $y'' = \alpha(\alpha-1) t^{\alpha-2}$

$$0 = a t^2 \alpha(\alpha-1) t^{\alpha-2} + c\, t^{\alpha} + b t \alpha t^{\alpha-1}$$

$$= t^{\alpha}\left[a\alpha^2 - a\alpha + b\alpha + c\right]$$

$$= t^{\alpha}\left[a\alpha^2 + (b-a)\alpha + c\right]$$

$$\alpha = \frac{-(b-a)}{2a} \pm \frac{\sqrt{(b-a)^2 - 4ac}}{2a}.$$

3 cases

1. $(b-a)^2 - 4ac > 0$

2. $(b-a)^2 - 4ac = 0$

3. $(b-a)^2 - 4ac < 0$.

Start

1. $\alpha = \alpha_+$ & α_-

$\alpha_+ = \frac{-(b-a)}{2a} + \sqrt{}$

$\alpha_- = \qquad -$

$y_1(t) = t^{\alpha_+}$ both solve ode

$y_2(t) = t^{\alpha_-}$

$\{y_1, y_2\}$ linearly independent.

<u>Case 2</u> $(b-a)^2 - 4ac = 0.$

$$y_1(t) = t^{\frac{-(b-a)}{2a}} = t^{\frac{a-b}{2a}} = t^{\alpha}$$

use method of reduction of order to find second

Try $y(t) = y_1(t)\,v(t).$

$$0 \overset{\text{want}}{=} at^2\left[\cancel{y_1''v} + \boxed{2y_1'v'} + \boxed{y_1 v''}\right]$$

$$+bt\left[\cancel{y_1'v} + \boxed{y_1 v'}\right]$$

$$+c\,\cancel{y_1 v}$$

$$= v\,\underbrace{\left[at^2 y_1'' + bt\,y_1' + c\,y_1\right]}_{=0}$$

$$+ at^2\,2y_1'v' + at^2\,y_1 v''$$

$$+bt\,y_1 v'$$

~~14~~ 15

so want

$$0 = a t^2 y_1 \gamma'' + a t^2 2 y_1' \gamma' + b t y_1 \gamma'$$

$$y_1 = t^\alpha = t^{\frac{a-b}{2a}}$$

$$0 = \cancel{t} t a y_1 \gamma'' + a t 2 y_1' \gamma' + b y_1 \gamma'$$

$$= t a t^\alpha \gamma'' + a t 2 \alpha t^{\alpha-1} \gamma' + b t^\alpha \gamma'$$

$$= t a \gamma'' + \underbrace{[2 a \alpha + b]}_{= a} \gamma'$$

$$= a [t \gamma'' + \gamma']$$

~~15~~ 16

$$0 = t\gamma'' + \gamma'$$

$$t \frac{d}{dt}\gamma' = -\gamma'$$

$$\frac{d\gamma'}{\gamma'} = \frac{-dt}{t}$$

$$\ln(\gamma') = -\ln(t) + c$$

$$\gamma' = \frac{c}{t}$$

$$\Rightarrow \gamma(t) = c\ln(t) + c_2$$

so $y(t) = t^\alpha \left[c\ln(t) + c_2\right]$

$$= c\, t^\alpha \ln(t) + c_2 t^\alpha$$

Also a solution.

$y_1(t) = t^\alpha \qquad y_2(t) = t^\alpha \ln(t).$

more correctly

$$y_1(t) = t^{\alpha} \qquad y_2(t) = t^{\alpha} \ln(|t|).$$

<u>case 3</u>

change of variables to make Euler constant-coeffecient.

~~$ay'' + by' +$~~

$$at^2 y'' + bty' + cy = 0$$

Try $y(t) = z(x)$ $\qquad x = \ln(t)$

$t = e^x$

look for ODE for z.

$$\frac{dy}{dt} = \frac{dz}{dx}\frac{dx}{dt} = \frac{dz}{dx}\frac{1}{t}$$

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$$y''(t) = \frac{d}{dt}\left[\frac{dz}{dx}\frac{1}{t}\right]$$

$$= \frac{-1}{t^2} z'(x) + \frac{1}{t}\frac{d}{dt}\left(\frac{dz}{dx}\right)$$

$$= -\frac{1}{t^2} z'(x) + \frac{1}{t}\left[\frac{d}{dx}\left(\frac{dz}{dx}\right)\frac{dx}{dt}\right]$$

$$= \frac{-1}{t^2} z'(x) + \frac{z''(x)}{t^2}$$

$$0 = a t^2 y'' + b t y' + c y$$

$$= a t^2\left[\frac{z''(x)}{t^2} - \frac{z'(x)}{t^2}\right] + b t\left[\frac{z'(x)}{t}\right] + c z$$

$$= a z''(x) + (b-a) z'(x) + c z(x)$$

characteristic eq.

$$ar^2 + (b-a)r + c = 0$$

$$r = -\frac{(b-a)}{2a} \pm \frac{\sqrt{(b-a)^2 - 4ac}}{2a}.$$

Case 1 $\quad (b-a)^2 - 4ac > 0.$

$r_1 \neq r_2$ roots.

$\cancel{z(x)=}$ $\quad z_i(x) = e^{r_i x}$

$$y_i(t) = z_i(x) = (e^x)^{r_i} = (t)^{r_i}$$

As expected.

Case 3 $\quad r = \lambda \pm i\omega.$

$$\lambda = -\frac{(b-a)}{2a} \qquad \mu = \frac{\sqrt{(b-a)^2 - 4ac}}{2a}.$$

$$z_1(x) = e^{\lambda x}\cos(\mu x)$$

$$z_2(x) = e^{\lambda x}\sin(\mu x).$$

$$y_1(t) = z_1(x) = (e^{x})^{\lambda}\cos(\mu x)$$

$$= t^{\lambda}\cos[\mu \ln(t)]$$

$$y_2(t) = |t|^{\lambda}\sin(\mu \ln(|t|)).$$

L1

Recall last class

$$t^2 10 y'' + t y' + y = t^2$$

got a particular solution of form $y_p(t) = c_1 t^4$

known.

Try solution $y(t) = C t^2$

$$t^2 \overset{\text{want}}{=} t^2 (10) 2C + t\, 2Ct + C t^2$$

$$\Rightarrow 1 = 23C \quad \Rightarrow \quad y(t) = \frac{1}{23} t^2$$

is a particular solution.

Our mistake.

For variation of parameter equation must be in standard form

$$y''(t) + p(t)y'(t) + q(t) = g(t).$$

So

$$\underbrace{y'' + \frac{1}{10t}y' + \frac{y}{10t^2} = \frac{1}{10} = f(t)}_{\text{Standard form}}$$

Now Apply variation of parameters

$\{y_1, y_2\}$ don't change, but

$f(t) = \frac{1}{10}$ not $f(t) = t^2$.

Existence of solutions to non linear odE.

Consider find a solution of

$$\begin{cases} y''(t) - 4y'(t) + 3y(t) = -(y(t))^2 & 0 \le t \le T \\ y(0) = 0 \\ y'(0) = 0. \end{cases} \qquad (0)$$

~~Try & Apply fixed point Argument to odE or try write it as integral equation.~~

__Two Approaches__

① write As integral equation.
& solve with fixed point eq.

② Solve with out using integral eq.

③ can write As system of 1st order ODE's.

Approach 2

consider the mapping $M(y) = \bar{y}$

by

$$\begin{cases} \bar{y}''(t) - 4(\bar{y})'(t) + 3\bar{y}(t) = -y(t)^2, \quad 0 \le t \le T \\ \bar{y}(0) = 0 \\ \bar{y}'(0) = 1. \end{cases}$$

Then if $M(y) = y$ then done.

So need to find fixed point of M.

Work on

$B_R = \{ y \in C[0,T] : \max_{0 \le t \le T} |y(t)| \le R \}$.

Note if $y \in C[0,T]$ then $-y(t)^2$ is continuous. NEED to be able to solve linear eq.

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$$\max_{0 \le t \le T} |y(t)| \le \frac{1}{2}\left|e^{3T} - e^{T}\right| \qquad (*)$$

$$+ C T R.$$

where C independent of T.

~~If y_1, y_2 solve (1) then~~

~~$y_2(t) - y_1(t) =$~~

If y_1 solves (1) with $f = f_1$

& y_2 solves (1) with $f = f_2$ then

$$y_2(t) - y_1(t) = e^{t} \int_0^t \frac{-e^{3\tau}}{2e^{4\tau}} \left[f_2(\tau) - f_1(\tau) \right] d\tau$$

$$+ e^{3t} \int_0^t \frac{e^{\tau}}{2e^{4\tau}} \left[f_2(\tau) - f_1(\tau) \right] d\tau$$

The linear theory

let $f \in C[0,T]$, where $0 < T \le 1$.

let
$$\begin{cases} y''(t) - 4y' + 3y = f(t), & 0 \le t \le T \\ y(0) = 0 \\ y'(0) = 1 \end{cases} \qquad (1)$$

has a solution y ξ

$$y(t) = \tfrac{1}{2} e^{3t} - \tfrac{1}{2} e^{t} + e^{t} \int_0^t \frac{-e^{3\tau}}{2e^{4\tau}} f(\tau)\, d\tau + e^{3t} \int_0^t \frac{e^{\tau}}{2e^{4\tau}} f(\tau)\, d\tau.$$

From this we see if $\sup_{0 \le t \le T} |f(t)| \le R$

then

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So

$$\boxed{\max_{0 \le t \le T} |y_2(t) - y_1(t)|} \qquad (**)$$

$$\le \max_{0 \le \tau \le T} |f_2(\tau) - f_1(\tau)| \left(C_2 T \right)$$

C_2 independent of f_i & $T \le 1$.

We now try to show

$M : B_R \to B_R$ a contraction

$B_R = \{ y \in C[0,T] : \max_{0 \le t \le T} |y(t)| \le R \}$.

<u>into</u> let $y \in B_R$, $M(y) = \bar{y}$ then

By ~~(**)~~ (*) with $f(t) = -y(t)^2$

get

$$\max_{0 \le t \le T} |\bar{y}| \le \frac{1}{2} |e^{3T} - e^{T}| + CT(R^2)$$

So for $M(B_R) \subset B_R$ ~~need~~ sufficient 18

$$\boxed{\tfrac{1}{2}\left|e^{3T}-e^{T}\right| + C T R^2 \le R} \quad \mathrm{I}$$

<u>contraction.</u> let $y_i \in B_R$ $\overline{y_i} = M(y_i)$

Then by $(**)$ have

$$\max_{0\le t\le T} \left|\overline{y_2}(t) - \overline{y_1}(t)\right| \le (C_2 T) \max_{0\le \tau\le T} \left| y_2(\tau)^2 - y_1(\tau)^2 \right|$$

$$\le (C_2 T) \max_{0\le \tau\le T} |y_2(\tau) + y_1(\tau)| \cdot \max_{0\le \tau\le T} |y_2(\tau) - y_1(\tau)|.$$

$$\le (C_2 T\, 2R) \max_{0\le t\le T} |y_2(t) - y_1(t)|$$

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so

$$\max_{0 \le t \le T} \left| M(y_2)(t) - M(y_1)(t) \right|$$

$$\le (C_2 T 2R) \max_{0 \le t \le T} \left| y_2(t) - y_1(t) \right|.$$

So need

$$\boxed{C_2 T 2R \le \frac{3}{4}} \quad \text{say} \quad \text{II}$$

To satisfy I & II fix $R > 0$

then pick ~~T > 0~~ $0 < T \le 1$ to satisfy I.

Then pick T smaller to satisfy

II.

<u>uniqueness</u>

let y_2 satisfy (0), set

$z(t) = y_2(t) - y_1(t)$, then

$$z''(t) - 4z'(t) + 3z(t) = -y_2(t)^2 + y_1(t)^2$$

$$= -(y_1 + y_2)(-y_1 + y_2)$$

$$= -(y_1 + y_2) z$$

set $c(t) = [y_1(t) - y_2(t)]$

then

$$\begin{cases} z''(t) - 4z'(t) + [3 + c(t)] z(t) = 0 & 0 \le t \le T \\ z(0) = z'(0) = 0. \end{cases} \quad (2)$$

want to show $z \equiv 0$.

If (3) has uniqe sol. (11)

ten $z \equiv 0$ (since clearly $z \equiv 0$ a solution).

let z_1, z_2 solve (3) & set

$$w(t) = \det\begin{pmatrix} z_1 & z_2 \\ z_1' & z_2' \end{pmatrix}$$

ten

$$w'(t) = 4w$$

$$\Rightarrow \begin{cases} w'(t) - 4w(t) = 0 & 0 \le t \le T \\ w(0) = z_1 z_2' - z_1' z_2 \big|_{t=0} = 0 \end{cases}$$

$\Rightarrow w(t) \equiv 0 \quad 0 \le t \le T.$

$\Rightarrow \{z_1, z_2\}$ Linearly dependent

on $[0, T]$