10

$f: \mathbb{R}^n \to \mathbb{R}^n$ smooth.

$$g(t) = f[x_1 + t(x_2 - x_1)]$$

$\vdots$

3.6 Method of undetermined co-efficients

let f, p & q given.

Goal Find y general solution of

$$y'' + p(t)y' + qy = f \qquad (1).$$

The Homogeneous eq. associated with (1) is

$$y'' + p(t)y' + qy = 0 \qquad (2).$$

Thm let y_p be a particular solution of (1) & $\{y_1, y_2\}$ fundamental set of solutions to (2) (ie y_i solves (2) & $\{y_1, y_2\}$ Linearly independent)

Then the general solution of (1) is

$$y = y_p + c_1 y_1 + c_2 y_2.$$

<u>proof</u>. Let Y solves (1) & y_p solves (1)

Then $Y - y_p$ solves (2) Hence $\exists\, c_1, c_2$

$$Y - y_p = c_1 y_1 + c_2 y_2.$$

Hence to find the general solution we need to just find a particular solution.

Method of undetermined co-efficient

(Guess method)

$y'' - 3y' - 4y = 3e^{2t}$

Find A particular solution.

Try $y(t) = ce^{2t}$

$y'' - 3y' - 4y = [c(4) - 3(2c) - 4c]e^{2t}$

$\Vert$ want

$3e^{2t}$

So want $c(4-6-4) = 3$

$c = -\frac{1}{2}$

So $y_p(t) = -\frac{1}{2}e^{2t}$ solves ODE.

Consider

$$y'' - 3y' - 4y = ~~\quad~~\ 3e^{\alpha t} \qquad \alpha \in \mathbb{R}.$$

Homogeneous

$$y'' - 3y' - 4y = 0$$

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0.$$

$$y_1(t) = e^{4t} \qquad y_2(t) = e^{-t}.$$

Provided $\alpha \neq 4$ or -1 try

$$y_p(t) = Ce^{\alpha t}.$$

$\underline{\alpha = 4}$ $\quad y'' - 3y' - 4y = 3e^{4t}$

~~[illegible]~~ Try $y_p(t) = e^{4t}C$ but

this solves Homogeneous problem.

5

Trick Try $y(t) = c\,t\,e^{4t}$

$= c\,t\,y_1(t) \qquad y_1 = e^{4t}$

$y' = c\,y_1 + c\,t\,y_1'$

$y'' = c\,y_1'\,2 + c\,t\,y_1''$

$3e^{4t} \overset{\text{want}}{=} 2c\,y_1' + c\,t\,y_1'' - 4\,c\,t\,y_1$

$-3(c\,y_1 + c\,t\,y_1')$

$= \cancel{ct\left[y_1'' - 3y_1' - 4y_1\right]}$

$= ct\left[\underbrace{y_1'' - 3y_1' - 4y_1}_{=0}\right]$

$+2c\,y_1' - 3c\,y_1$

$= c(2y_1' - y_1) =$

$= c\left(2(4) - 1\right]e^{4t}$

want $3 = C(8-1)$

$C = \frac{3}{7}$ so particular sol.

$y_p(t) = \frac{3}{7} t e^{4t}$.

Ex Find the general solution

of $y''(t) + 4y(t) = \sin(3t)$

Solution. check Homogenous.

$y'' + 4y = 0$

$y_1(t) = \sin(2t)$ $y_2(t) = \cos(2t)$.

Guess $y(t) = A\sin(3t) + B\cos(3t)$.

since neither solves Homogenous

this is a good guess.

Find the general solution of

$$y'' - 3y' - 4y = 2\sin(2t)$$

Homogeneous

$y_1(t) = e^{-t}$ $\quad y_2(t) = e^{4t}$

Try $y(t) = A\sin(t2) + B\cos(2t)$.

$$y' = 2A\cos(2t) - 2B\sin(2t)$$

$$y'' = -4A\sin(2t) - 4B\cos(2t).$$

$$2\sin(2t) \overset{\text{want}}{=} y'' - 3y' - 4y$$

$$= (-8A + 6B)\sin(2t)$$

$$+ (-6A - 8B)\cos(2t)$$

so want
$$-8A + 6B = 2$$
$$-6A - 8B = 0$$

8

$$\begin{pmatrix} -8 & 6 \\ -6 & -8 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

invertible matrix.

let A, B solve Above Hence

general solution

$$y(t) = A\sin(2t) + B\cos(2t) + c_1 e^{-t} + c_2 e^{4t}$$

free.

3.7 Variation of parameters

[9

start here

let $\{y_1, y_2\}$ solve

$$y'' + p(t)y' + q(t)y = 0.$$

The method gives you a particular solution of

$$y'' + py' + qy = f(t).$$

Look for solution of the form

$$y = u(t)y_1(t) + v(t)y_2(t)$$

$$y' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

$$y'' = u''y_1 + 2u'y_1' \cancel{+ u'y_1'}$$

$$+ uy_1'' + v''y_2 + 2v'y_2' + vy_2''$$

So want

$$f(t) \overset{\text{want}}{=} u''y_1 + 2u'y_1' + \cancel{uy_1''} + v''y_2 + 2v'y_2' + \cancel{vy_2''} + p\left(u'y_1 + \cancel{uy_1'} + v'y_2 + \cancel{vy_2'}\right) + q\left[\cancel{uy_1} + \cancel{vy_2}\right]$$

$$= u\underbrace{\left[y_1'' + py_1' + qy_1\right]}_{=0} + v\underbrace{\left[y_2'' + py_2' + qy_2\right]}_{=0} + u''y_1 + 2u'y_1' + v''y_2 + 2v'y_2' + pu'y_1 + pv'y_2$$

lets now look for u & v s.t.

$\cancel{u'y_1 + v'y_2}$

$$u'y_1 + v'y_2 = 0 \quad * \quad \text{Assumption}$$

$$u''y_1 + u'y_1' + v''y_2 + v'y_2' = 0$$

then

$$f(t) \overset{\text{want}}{=} u''y_1 + 2u'y_1' + v''y_2 + 2v'y_2'$$

$$= -u'y_1' - v'y_2' + 2u'y_1' + 2v'y_2'$$

$$= u'y_1' + v'y_2'$$

So we need to find u & v s.t.

$$u' y_1' + v' y_2' = f(t)$$
$$u' y_1 + v' y_2 = 0$$

$$y_1 u' + y_2 v' = 0$$
$$y_1' u' + y_2' v' = f$$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

<u>Recall</u> $w(t) = w(y_1, y_2)(t) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$

rule from lin Alg

$$u' = \frac{\det\begin{pmatrix} 0 & y_2 \\ f & y_2' \end{pmatrix}}{W} = -\frac{y_2 f}{W}.$$

$$v' = \frac{\det\begin{pmatrix} y_1 & 0 \\ y_1' & f \end{pmatrix}}{W} = \frac{y_1 f}{W}.$$