Ex $\quad (x^2-3y^2)y' = -2xy-6 \qquad (0)$

$$\underbrace{(2xy+6)}_{M} + \underbrace{(x^2-3y^2)}_{N}y' = 0 \qquad (1)$$

$N_x = 2x$

$M_y = 2x$

so (1) is exact.

$f_x = M = 2xy+6$

$f = x^2y + 6x + h(y)$

$f_y = x^2 + h'(y)$

$x^2 + h'(y) = f_y = N = x^2 - 3y^2$

$\Rightarrow h'(y) = -3y^2 \Rightarrow h(y) = -y^3$

so $C = x^2y + 6x - y^3$

An implicit solution of (0) for each C.

Comment

$\psi_x = M \qquad \psi_y = N$

we started by integrating $\psi_x = M$ in x; If you cannot then start with $\psi_y = N$ & integrate in y.

---

integrating factors (to make Exact).

Ex $\underbrace{(3xy+y^2)}_{= M} + \underbrace{(x^2+xy)}_{N} y' = 0 \qquad (2)$

$M_y = 3x+2y \qquad N_x = 2x+y$

$M_y \neq N_x$ so not exact.

Integrating factors (to make ODE Exact)

Consider

$$M + N y' = 0 \qquad (2)$$

with $M_y \neq N_x$ (not exact).

Multiply by $\mu = \mu(x,y)$ (to be determined)

$$\bar{M} + \bar{N} y' = 0 \qquad (3)$$

$$\bar{M} = M\mu \qquad \bar{N} = N\mu$$

& note if $\mu \neq 0$ (on any interval) then solutions of (3) & (2) should be the same.

(3) is exact if

$$(M\mu)_y = (N\mu)_x = N_x \mu + N \mu_x$$

$$\| $$

$M_y \mu + M \mu_y$.

(In general).

Suppose $\mu = \mu(x)$ then

$$M_y \mu = N_x \mu + N \mu_x \qquad (\mu = \mu(x))$$

$$\cancel{M_y \mu + M \mu_y = N_x \mu} \qquad \cancel{(\mu = \mu(x))}$$

$$\frac{M_y - N_x}{N} = \frac{\mu_x}{\mu} \qquad \mu = \mu(x)$$

If $\mu = \mu(y)$ then

$$M_y \mu + M \mu_y = N_x \mu.$$

$$\frac{\mu_y}{\mu} = \frac{(N_x - M_y)}{M}$$

~~so there will be~~

If $\dfrac{M_y - N_x}{N}$ depends just on x

try $\mu = \mu(x)$.

If $\dfrac{N_x - M_y}{M}$ depends on just y

try $\mu = \mu(y)$.

Ex Consider

$$\underbrace{(3xy+y^2)}_{= M} + \underbrace{(x^2+xy)}_{= N} y' = 0$$

$M_y \neq N_x$ so not exact.

$$M_y - N_x = 3x + 2y - (2x+y)$$
$$= 3x - 2x + y = x + y$$

$$\frac{M_y - N_x}{N} = \frac{x+y}{x^2+xy} = \frac{x+y}{x(x+y)} = \frac{1}{x}$$

So there is a $\mu = \mu(x)$.

$$\frac{1}{x} = \left(\frac{d\mu}{dx}\right)\frac{1}{\mu}.$$

$$\frac{1}{x}\,dx = \frac{1}{\mu}\,d\mu \qquad \text{separable}$$

$\ln(x) = \ln(\mu) + c$

$\boxed{\mu = \cancel{}\ c x}$ take $c = 1$.

$x(3xy + y^2) + x(x^2 + xy)y' = 0$ is exact

$\bar{M} = 3x^2y + xy^2$ $\quad \bar{N} = x^3 + x^2y$

$\psi_x = \bar{M}$

$\psi = x^3y + \frac{x^2}{2}y^2 + h(y)$.

$x^3 + x^2y = \bar{N} = \psi_y = x^3 + x^2y + h'(y)$

$\Rightarrow h'(y) = 0$ $\quad \Rightarrow h(y) = c = 0$

so $x^3y + \frac{x^2}{2}y^2 = c$ implicit solution of (3) & hence of (2).

2-8 Existence of solution of initial value problem. Find $\delta > 0$

$$\begin{cases} y'(t) = f[t, y(t)] & t \in [0, \delta) \\ y(0) = y_0 \end{cases} \quad (1)$$

has a solution. Assume $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous. Suppose we have A solution of (1); integrate in t

$$y(t) - y(0) = \int_0^t f(\tau, y(\tau))\, d\tau$$

Or

$$y(t) = \cancel{\ } \ y_0 + \int_0^t f(\tau, y(\tau))\, d\tau \quad (2)$$

(2) is called the integral equation associated with (1). So if y solves (1) then it solves (2). Now suppose $y \in C[0,d)$ solves (2) all $0 \le t < d$ then y is differentiable on $[0,d)$ hence we can take derivative of (2) to see y solves (1).

So we will find a solution of (2), by fixed point.

Special case.

$f = f(y)$ & f Lipshitz, $\exists C > 0$

$$|f(x) - f(y)| \le C\,|x-y| \quad \forall x, y \in \mathbb{R}.$$

For $0 < T$ (small) define

$M_{\#} : C[0,T] \to C[0,T]$ by

$$M_{\#}(y)(t) := y_0 + \int_0^t f(y(\tau))\, d\tau.$$

Start

Claim For $T > 0$ small

$M : C[0,T] \to C[0,T]$ is a contraction & hence $\exists!\, y_0 \in C[0,T]$

s.t. $M(y) = y$ in $C[0,T]$

ie $M(y)(t) = y(t) \quad \forall \; t \in [0,T]$

But this y exactly solves (2).

<u>proof of claim</u>

want to show

$d(M(\bar{y}), M(y)) \le \gamma\, d(\bar{y}, y)$. $\gamma < 1$

But recall

$$d(\bar{y}, y) = \|\bar{y} - y\|_{C[0,T]}$$

$$= \sup_{0 \le t \le T} |\bar{y}(t) - y(t)|.$$

let $\bar{y}, y \in C[0,T]$.

$$M(\bar{y})(t) = y_0 + \int_0^t f[\bar{y}(\tau)]\, d\tau$$

$$M(y)(t) = y_0 + \int_0^t f(y(\tau))\, d\tau$$

so let $0 \le t \le T$

$$m(\bar{y})(t) - m(y)(t) = \int_0^t f(\bar{y}(\tau)) - f(y(\tau))\, d\tau$$

$$|m(\bar{y})(t) - m(y)(t)| \le \int_0^t |f(\bar{y}(\tau)) - f(y(\tau))|\, d\tau$$

$$\le \int_0^t c\, |\bar{y}(\tau) - y(\tau)|\, d\tau$$

$$\le c \sup_{0 \le \tau \le t} |\bar{y}(\tau) - y(\tau)|\, t$$

$$\le cT \sup_{0 \le \tau \le T} |\bar{y} - y|$$

$$\sup_{0 \le t \le T} |m(\bar{y})(t) - m(y)(t)| \le cT \sup_{0 \le \tau \le T} |\bar{y} - y|$$

or

$$\| M(\bar{y}) - M(y) \|_{C[0,T]} \le (CT) \| \bar{y} - y \|_{C[0,T]}.$$

So if $CT < 1$ then

$M : C[0,T] \longrightarrow C[0,T]$ is

a contraction. ~~[illegible]~~

Same proof works if $\exists c > 0$

$|f(t,x) - f(t,y)| \le c\,|x-y|$

$\forall\, x,y \in \mathbb{R}$ $\forall\, t \in \mathbb{R}$.

---

case of f Locally Lipshitz.

(do special case) $f = f(x)$, but

same proof works if $f = f(t,x)$,

if Locally Lipshitz in x.

Assumption on f

$\forall R > 0$ $\exists C_R > 0$

$|f(x) - f(y)| \le C_R\,|x-y|$ $\forall\, |x|, |y| \le R$.

$$\left(\begin{array}{ll} f(x) = x^2, \ldots & \checkmark \\ f(x) = \sqrt{|x|} & \text{NO} \end{array}\right)$$

Special case. $f(x) = x^2$.

$$y(t) = y_0 + \int_0^t (y(\tau))^2 d\tau \qquad 0 \le t \le T \quad (2)$$

find $y \in C(I)$ $\quad I = [0, T]$ solves (2).

Let $\bar{y}, y \in C(I)$

$$m(\bar{y})(t) - m(y)(t) = \int_0^t (\bar{y}(\tau))^2 - (y(\tau))^2 \, d\tau$$

$$= \int_0^t (\bar{y}(\tau) + y(\tau))(\bar{y}(\tau) - y(\tau)) \, d\tau.$$

$$|m(\bar{y})(t) - m(y)(t)| \le \left(T \sup_{0 \le t \le T} |\bar{y}(t) + y(t)|\right) \|\bar{y} - y\|_{C(I)}$$

$$\|m(\bar{y}) - m(y)\|_{C(I)} \le \boxed{T \sup_{0 \le t \le T} |\bar{y} + y|} \; \|\bar{y} - y\|_{C(I)}$$

want

$$T \sup_{0 \le t \le T} |\bar{y} + y| < 1$$

But clearly this impossible for all $\bar{y}, y \in C(I)$.

<u>Idea</u>: Don't work on all of $C(I)$ but rather a subset.

~~Without loss of generality take~~

~~$y_0 = 0$~~

~~$y'(t) = f[y(t)]$~~

~~$y(0) = y_0$.~~

For $R > 0$ set

$$B_R(y_0) = \{ y \in C(I) : \|y - y_0\|_{C(I)} \le R \}.$$

$$d(\bar{y}, y) = \|\bar{y} - y\|_{C(I)}$$

$(B_R(y_0), d)$ complete metric space.

<u>Claim</u> For $R>0$ suitably chosen & $T>0$ small

$M: B_R \to B_R$ a contraction.

① contraction

② into $M(B_R) \subset B_R$.

① let $\bar{y}, y \in B_R(y_0)$

$$|M(\bar{y})(t) - M(y)(t)| \leq \int_0^t |\bar{y}(\tau) - y(\tau)|\, |\bar{y}(\tau) + y(\tau)|\, d\tau$$

$$\leq \sup_{0 \leq t \leq T} |\bar{y}(t) - y(t)|\; T \sup_{0 \leq t \leq T} \left[|\bar{y}| + |y| \right]$$

$$\leq \|\bar{y} - y\|_{C(I)}\, T \sup_{0 \leq t \leq T} \left[|\bar{y}(t)| + |y(t)| \right]$$

$|y(t) - y_0| \leq R$

$\therefore\ |y(t)| \leq |y_0| + R$.

$$\|m(\tilde{y}) - m(y)\|_{C(I)} \le T\,2\,[|y_0| + R]\,\|\tilde{y} - y\|_{C(I)}$$

want $\quad 2T\,[|y_0| + R] \le \frac{3}{4} \qquad (*)$

~~into~~ $m(B_R(y_0)) \subset B_R(y_0)$.

$y \in B_R(y_0)$

$$|m(y)(t)| \le |y_0| + \int_0^t |y(\tau)|^2\, d\tau$$

$$\le |y_0| + T\ |y_0|$$

$$\le |y_0| + T\ (|y_0| + R)^2$$

so

<u>into</u> $y \in B_R(y_0)$

$$M(y)(t) = y_0 + \int_0^t (y(\tau))^2 \, d\tau.$$

$$|M(y)(t) - y_0| \le \int_0^t |y(\tau)|^2 \, d\tau$$

$$\le T\left[|y_0| + R\right]^2$$

$$\therefore \sup_{0 \le t \le T} |M(y)(t) - y_0| \le T(|y_0| + R)^2$$

so $M(B_R(y_0)) \subset B_R(y_0)$ if

$$\boxed{T(|y_0| + R)^2 \le R} \quad (**)$$

Fix $R > 0$ then take T small

to satisfy $(*)$ & $(**)$.