$\underline{1.2}$ & $\underline{2.8}$) (Existence & uniqueness) L'

let $f : \mathbb{R}^2 \to \mathbb{R}$ s.t. $\forall R > 0$

$\exists C_R > 0$ s.t.

$|f'(x,y)| + |f(x,y)| \le C_R$ (conditi (A)

$\forall |x| \le R$ & $|y| \le R$.

Th^m let f satisfy condition (A) &

consider

$$\begin{cases} y'(x) = f(x, y(x)) & x \in I \text{ open interval } (1) \\ y(x_0) = y_0 & x_0 \in I. \end{cases}$$

Then there is at most one solution of (1).

proof copy proof of linear case. L2

let y_1 & y_2 solve (1) & let

$$z(x) = y_1(x) - y_2(x)$$ so

$$z'(x) = y_1'(x) - y_2'(x)$$
$$= f[x, y_1(x)] - f(x, y_2(x))$$

need to "make linear".

$$z'(x) = \left[\frac{f[x, y_1(x)] - f(x, y_2(x))}{y_1(x) - y_2(x)}\right] z(x)$$

$$= c(x)\, z(x)$$

$$c(x) := \frac{f(x, y_1(x)) - f(x, y_2(x))}{y_1 - y_2}.$$

Think of $I = (x_0 - \delta_8, x_0 + \delta_8)$

So maybe y blows up at end points but for $0 < \bar{\delta} < \delta$ y (& hence y') is Bounded on $J = (x_0 - \bar{\delta}, x_0 + \bar{\delta})$ & hence

$c(x)$ is bounded on J.

So

$$\begin{cases} z'(x) - c(x) z(x) = 0 & x \in I \\ z(x_0) = 0 \end{cases}$$

$$\mu(x) = e^{\int_{x_0}^{x} -c(z)\,dz}$$ Bounded on J.

$$\frac{d}{dx}\left[\mu(x)\, z(x)\right] = 0$$

$$\Rightarrow \mu(x) z(x) - \mu(0) \overset{0}{\cancel{z(0)}} = 0 \quad \text{on } I$$

$u(x) z(x) = 0$ on I &

$u \neq 0$ on J $\Rightarrow$ $z \equiv 0$ on J

~~but~~ but $0 < \delta < d$ arbitrary

$\Rightarrow$ $z \equiv 0$ on I.

Ex (non-uniqueness).

Aside look for solution of

$$\begin{cases} y' = y^{\frac{1}{3}} & (2) \\ y(0) = 0. \end{cases}$$

(of course $y \equiv 0$ a solution)

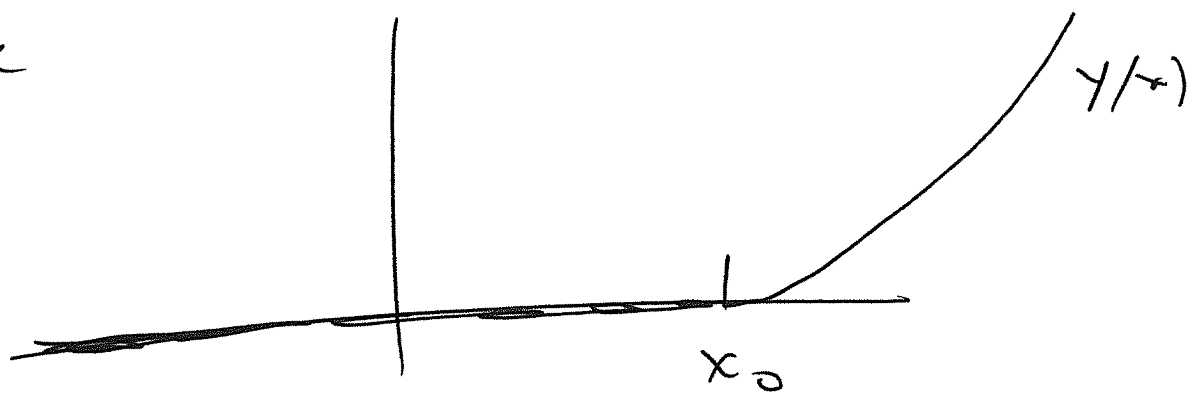
Try $y(x) = C x^t$ $C, t > 0$ to be determined.

Then we see $y(x) = \left(\frac{2}{3} x\right)^{\frac{3}{2}}$ solves (2). note its not defined for $x < 0$.

put $y(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \left(\frac{2}{3}x\right)^{\frac{3}{2}} & \text{if } x > 0 \end{cases}$

then y' is continuous & y solves (2); but $y \equiv 0$ Also solves (2); Hence no unique solution.

If fact can pick $x_0 \in \mathbb{R}$ & have



$$y(x) = \begin{cases} 0 & \text{if } x \le x_0 \\ \left(\frac{2}{3}(x - x_0)\right)^{\frac{3}{2}} & \text{if } x > x_0 \end{cases}$$

solves (2) Any $x_0 > 0$.

Note (2) is of form

~~[illegible]~~ $y' = f(x,y)$

$f(x,y) = y^{\frac{1}{3}}$ ~~is~~ ~~NOT~~ ~~locally~~ Does not satisfy conditions

~~Lipschitz~~ A.

<u>Thm 2.4.1</u> (uniqueness)

$$\begin{cases} y' + p(t)\, y = g \\ y(t_0) = y_0 . \end{cases} \qquad (4)$$

Let $I = (\alpha, \beta)$ & p, g continuous on I (maybe blow up at end pts) & $t_0 \in I$. Then $\exists!$ solution of (4) on I.

Ex 1 pg 67.

Use Thm 2.4.1 to find interval in which the ~~Initi~~ (IVP) (initial value problem) has a unique solution

$$\begin{cases} t y' + 2y = 4t^2 \\ y(1) = 2. \end{cases}$$

Also find the solution.

$$y(t) = t^2 + \frac{1}{t^2}$$