2.2 (Boyce - Diprima)

Seperable equation.

Given a first order ODE

$\frac{dy}{dx} = f(x,y)$ we say

its seperable if

$f(x,y) = g(x)\,h(y)$ since

& hence

$\frac{dy}{dx} = g(x)\,h(y)$

can "seperate"

$\frac{1}{h(y)}dy = g(x)dx$ & just

integrate $\int \frac{1}{h(y)}dy = \int g(x)dx$

Is this fully rigourous?
(Doesn't matter, we get formula in end we can check),

The resulting formula for the (a) solution will generally be in implicit form (IE $F(x,y)=c$) & not explicit $y = \boxed{}$.

Ex Consider finding a solution of

$$\frac{dy}{dx} = \frac{x}{y^2} \qquad (1)$$

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{x}{y^2} & (2) \\ y(0) = 3 \end{cases}$$

Solution.

$\frac{dy}{dx} = \frac{x}{y^2}$ $\quad$ $y^2\,dy = x\,dx$ $\quad$ (so separable)

$$\int y^2\,dy = \int x\,dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} + C$$

$$\Rightarrow \boxed{y = \left(\frac{3}{2}x^2 + C\right)^{\frac{1}{3}}}$$

this solves

so for ANY $C \in \mathbb{R}$

(1).

lets pick C s.t. $y(0) = 3$.

$$3 \overset{\text{want}}{=} y(0) = \left(\frac{3}{2}0^2 + C\right)^{\frac{1}{3}}$$

$$\therefore C = 27$$

So $y(x) = \left(\frac{3}{2}x^2 + 27\right)^{\frac{1}{3}}$ solves (2).

<u>note</u> For this example we could isolate y; since it has $y^3 = f(x)$.

If it has $y^2 = f(x)$ then one must choose what branch of square root.

Ex Find a (the) solution of

$$\frac{dy}{dx} = -\frac{x}{y} \quad (1)$$

$$\begin{cases} \frac{dy}{dx} = -\frac{x}{y} \\ y(0) = 1 \end{cases} \quad (2)$$

$$\begin{cases} \frac{dy}{dx} = -\frac{x}{y} \\ y(0) = -1 \end{cases} \quad (3)$$

$$\begin{cases} \frac{dy}{dx} = -\frac{x}{y} \\ y(1) = 0 \end{cases} \quad (4)$$

Solution.

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y\,dy = -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\boxed{y^2 + x^2 = c}$$

implicit solution for (1); cannot isolate y in general here;

ie

$$y = \sqrt{c - x^2}$$

$$\text{or } y = -\sqrt{c - x^2}.$$

So $x^2+y^2=c$ is an implicit solution for (1).

Solve (2) pick c

want $y(0)=1$ $\quad 0^2+(1)^2=c$

$\Rightarrow c=1.$

So $x^2+y^2=1$ implicit solution of (2). Lets find explicit solution

$\boxed{y=\pm\sqrt{1-x^2}}$ but we want

$y(0)=1 \Rightarrow$ $y(x)=\sqrt{1-x^2}$ explicit solution of (2) on the interval $(-1,1)$.

This domain is the largest domain possible.

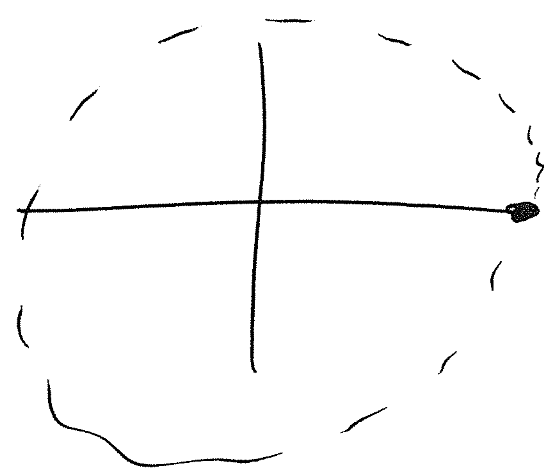
For (3) an implicit solution is $y^2+x^2=1$; the explicit is $y(x) = -\sqrt{1-x^2}$ & note the maximum interval of existence is $(-1,1)$.

For (4) $x^2+y^2=C$ implicit solution

find C; $y(1)=0$ $\quad C = 1^2+0^2 = 1$

$x^2+y^2=1.$

~~Note~~



If we want to find an explicit solution $y=y(x)$

& lets Allow $y'(1) = \pm\infty$

then note

$y(x) = \sqrt{1-x^2}$ & $y(x) = -\sqrt{1-x^2}$

is a solution of ~~the~~ (4)

on $[-1,1)$ with correct initial condition at $x=1$. (note solution not unique

Aside Implicit function theorem

let $F: \mathbb{R}^2 \longrightarrow \mathbb{R}$ smooth

$F(x_0, y_0) = 0$ & $F_y(x_0, y_0) \neq 0$.

Then $\exists$ some $\delta > 0$ &

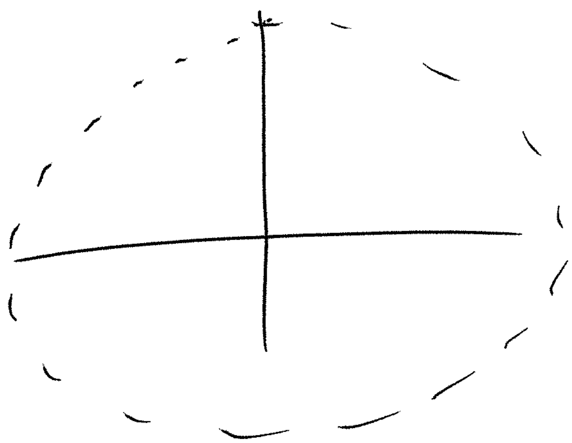
$x \mapsto y(x)$ continuous on $(x_0-\delta, x_0+\delta)$

s.t. $y(x_0) = y_0$ &

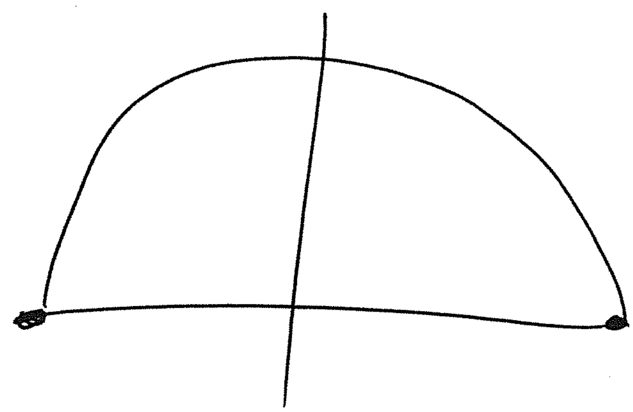
$0 = F(x, y(x))$ $\forall x \in (x_0-\delta, x_0+\delta)$.

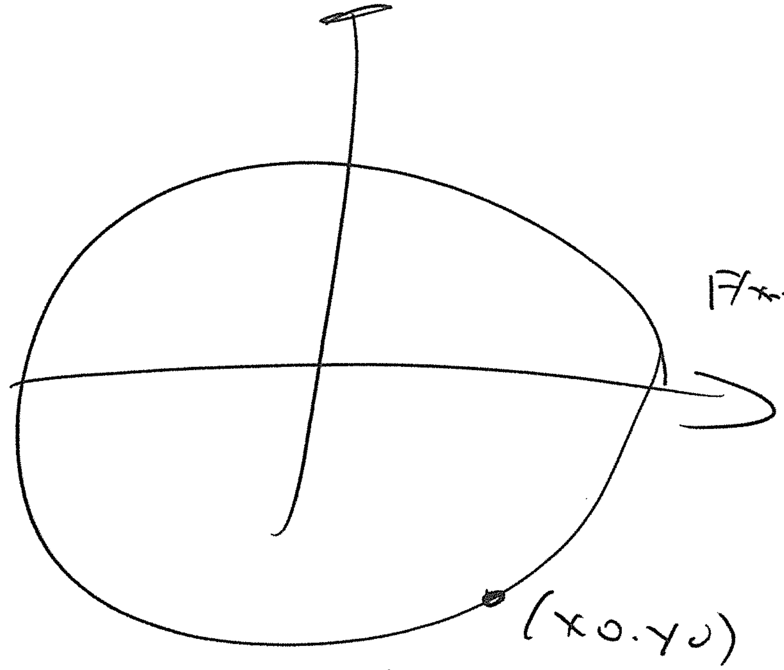
Intutively it says if you have implicit ~~graph~~ curve you can generally write the curve near a point explicitly.

Ex $x^2+y^2=1$ $(x_0, y_0) = (0,1)$



take $y = \sqrt{1-x^2}$





$F(x,y) = x^2+y^2-1=0$

$\partial_y F(x_0, y_0) = 2y_0 \neq 0$

So can write curve explicity

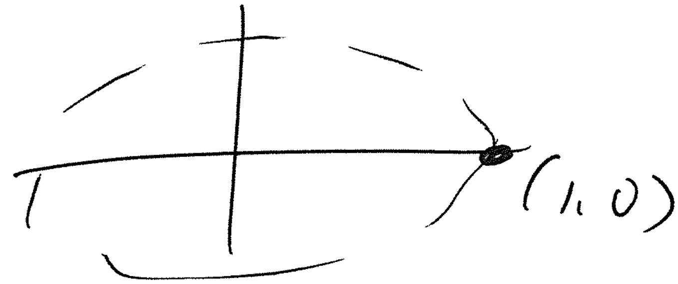
$y = -\sqrt{1-x^2}$ soln

$F(x, y(x)) = 0 \qquad y(x_0) = y_0$

$-1 < x < 1$.

At $(1,0)$ note $\partial_y F(1,0) = 2y\big|_{y=0} = 0$

can't apply thm.



No way to write curve explicitly near ~~xxx~~ $(1,0)$.

proof of I.F.T.

let $F(x_0, y_0) = 0$ $\quad F_y(x_0, y_0) = 0$

Find y $\quad F(x, y(x)) = 0$ $\quad |x - x_0| < d$

$y(x_0) = y_0.$

want

$$0 = F(x, y) = F(x, y$$

$$= F(x, y) - F(x_0, y)$$

$$+ F(x_0, y)$$

$$F(x_0, y) = F(x_0, y_0) + F_y(x_0, y_0)(y - y_0)$$

$$+ g(y) \qquad \exists\, c > 0$$

$$|g(y)| \le C|y|^2 \qquad \forall\, |y - y_0| \le 1$$

Find $y(x)$

$$0 \overset{\text{want}}{=} F(x,y) - F(x_0,y) + \cancel{F(x_0,y_0)}^{\,0} + F_y(x_0,y_0)(y-y_0) + g(y)$$

$$F_y(x_0,y_0)(y-y_0) = F(x_0,y) - F(x,y) - g(y)$$

WOLOG $\quad F_y(x_0,y_0) = 1$

$$y - y_0 = F(x_0,y) - F(x,y) - g(y)$$

want

$$y(x) = y_0 + F(x_0,y(x)) - F(x,y(x)) - g(y). \quad (1)$$

Given ~~$y: [x_0 - d, x_0 + d]$.~~

$I_d = [x_0 - d, x_0 + d]$

Given $y \in C(I_d)$ define

$$T(y)(x) := y_0 + F[x_0, y(x)] - F(x, y(x)) - g(y(x)).$$

If we can find fixed point of y on $C(I_d)$ then

$T(y) = y$ ie $T(y)(x) = y(x) \;\; \forall x \in I_d$

Hence $y(x)$ solves (1) & we are done. Idea is Apply Banach fixed pt thm.