L5

$$\begin{cases} x'(t) + b(t)\,x(t) = f(t) \\ x(0) = x_0 \end{cases} \qquad (2).$$

Typos from HW #1

#2 $e(x,y) = |x-y| + \left|\frac{1}{x} - \frac{1}{y}\right|$

show metric on $(0,\infty)$.

#3 Hint show $\exists$ fn ...

should be $\exists$ fnc ...

L6

Start Here.

$$\mu(t) = e^{\int_0^t b(\tau)\,d\tau}.$$

& take $t_0 = 0$

then

$$\mu(t)x(t) - \overset{1}{\mu(0)}\,\overset{x_0}{x(0)} = \int_0^t f(\tau)\mu(\tau)\,d\tau$$

$$x(t) = \frac{x_0}{\mu(t)} + \frac{1}{\mu(t)}\int_0^t f(\tau)\mu(\tau)\,d\tau.$$

Is this the only solution of (D). Let $y(t)$ Also solve (D) & since linear it makes sense to consider difference

$z(t) = x(t) - y(t)$.

$z'(t) + b(t) z(t)$

$= x' - y' + bx - by$

$= x' + bx - [y' + by] = f - f = 0$

$z(0) = x(0) - y(0) = 0$

so

$$\begin{cases} z'(t) + b(t) z(t) = 0 & t \in \mathbb{R} \\ z(0) = 0. \end{cases} \quad (3)$$

Of course $z \equiv 0$ is a solution.

But ~~Ace~~ is $z \equiv 0$ the only solution? If it is then

$y(t) \equiv x(t) \Rightarrow x(t)$ the unique solution of (2)

Use integrating factor again

$$\mu(t) = e^{\int_0^t b(\tau)\,d\tau}$$

$$\underbrace{z'\mu + bz\mu}_{\displaystyle = \frac{d}{dt}\left[z(t)\mu(t)\right]} = 0$$

$$\Rightarrow z(t)\mu(t) - \underbrace{z(0)}_{0}\,\underbrace{\mu(0)}_{1} = 0$$

$$\Rightarrow z(t)\mu(t) \equiv 0$$

$$\Rightarrow \boxed{z \equiv 0}$$

Consider $y'(t) + 4y(t) = 1$ (4) no initial condition.

$$\mu(t) = e^{\int_0^t 4\,d\tau} = e^{4t}$$

$$\underbrace{e^{4t} y'(t) + 4e^{4t} y}_{\| \atop \frac{d}{dt}\left(e^{4t} y(t)\right)} = e^{4t}$$

$$e^{4t} y(t) = \frac{1}{4} e^{4t} + C.$$

$$\boxed{y(t) = \frac{1}{4} + C e^{-4t} \qquad (5)}$$

This solves (4) for ANY CER. note eq is 1ST order & one free parameter (C). This is what will always happen. We know justify All solutions of (4) are given in (5). We call (5) the *general solution* of (4).

TP

[illegible]

let $y(t)$ solve (4) (Some solution)

& consider the Initial value problem

$$\begin{cases} x'(t) + 4x(t) = 1 & t \in \mathbb{R} \\ x(0) = y(0). \end{cases} \qquad (6)$$

We Already know (6) has a unique solution given by y; but instead lets use the formula.

Consider looking for a solution of the form

$x(t) = \frac{1}{4} + Ce^{-4t}$ some C.

[11]

we know for all c it solves the ODE; just need to solve the initial condition.

want $y(0) = x(0) = \frac{1}{4} + c e^{0}$

$\therefore \ c = y(0) - \frac{1}{4}$, so

$x(t) = \frac{1}{4} + (y(0) - \frac{1}{4}) e^{-4t}$ solves

(6) & hence

$y(t) = \frac{1}{4} + [y(0) - \frac{1}{4}] e^{-4t}$.

Hence <u>ALL</u> solutions of (4) are given by (5).

Recipe for first order linear

Re-arrange to

interval

$$y'(t) + b(t)y(t) = f(t) \qquad t \in I$$

$$t_0 \in I$$

$$\mu(t) = e^{\int b(\tau)d\tau}$$

multiply by $\mu(t)$

$$\underbrace{\mu(t)y' + b\mu y}_{\displaystyle \frac{d}{dt}(\mu y)} = \mu f$$

Now integrate.