proof. **Start Here** IS 19

Given x set

$T^2(x) = T[T(x)]$ $\qquad$ $T^n(x) = T[T(\cdots T(x)]$.

Lemma Given x, y, let $n > m$.

$d(T^n(x), T^m(y))$

$= d[T(T^{n-1}(x)), T[T^{m-1}(y)]]$

$\leq \gamma\ d(T^{n-1}(x), T^{m-1}(y))$

$\leq \gamma^2\ d(T^{n-2}(x), T^{m-2}(y))$.

$\vdots$

$\leq \gamma^m\ d(T^{n-m}(x), y)$.

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idea Fix $x_0 \in X$ & set

$x_1 = T(x_0)$ $\qquad$ show

$x_2 = T(x_1) = T^2(x_0)$ $\qquad$ $(x_n)_n$ is cauchy.

$x_n = T^n(x_0)$.

let $n > m$.

$$d(x_n, x_m) = d(T^n(x_0), T^m(x_0))$$

$$\leq \gamma^m \, d(T^{n-m}(x_0), x_0).$$

estimate $d(T^{n-m}(x_0), x_0)$.

Special case. $n-m=3$.

$$d(T^3(x_0), x_0) \leq d(T^3(x_0), T^2(x_0)) + d(T^2(x_0), x_0)$$

$$\leq \gamma^2 d(T(x_0), x_0) + d(T^2(x_0), T(x_0))$$

$$+ d(T(x_0), x_0)$$

$$\leq \gamma^2 d(T(x_0), x_0) + \gamma \, d(T(x_0), x_0)$$

$$+ d(T(x_0), x_0)$$

$$= (d(T(x_0), x_0)) \left[\gamma^2 + \gamma + 1 \right]$$

$$d(T^{n-m}(x_0), x_0) \leq \boxed{d(T(x_0), x_0)} \left[\sum_{k=0}^{n-m-1} \gamma^k \right]$$

so

$$d(x_n, x_m) \le \gamma^m d(T^{n-m}(x_0), x_0)$$

$$\le \gamma^m d(T(x_0), x_0) \sum_{k=0}^{n-m-1} \gamma^k$$

$$\le \gamma^m d(T(x_0), x_0) \sum_{k=0}^{\infty} \gamma^k.$$

$\longrightarrow 0$ as $m \to \infty$.

so $(x_n)_n$ is cauchy, $\therefore \exists x \in X$

~~st~~ ~~&~~ $x_n \to x$ in X.

But $x_n = T(x_{n-1})$

so by continuity

$$\underset{\substack{\downarrow \\ x}}{x_n} = T(x_{n-1}) \longrightarrow T(x)$$

so $T(x) = x$.

Uniqueness

Assume $\exists\, x \,\&\, z \in X$

$T(x) = x \quad \& \quad T(z) = z.$

Then

$$d(x, z) = d(T(x), T(z)) \le \gamma\, d(x, z)$$

$$\Rightarrow \underbrace{(1-\gamma)}_{>0}\, d(x, z) \le 0$$

$$\Rightarrow d(x, z) = 0 \Rightarrow x = z.$$

ODE's.

Given an ODE $x=x(t)$

$$F(t, x, x'', x''', x^{(n)}) = 0$$

we say the order of the ode is the order of the highest derivative appearing in eq.

Ex ① $x''(t) + \sin(x(t)) = f(t)$ 2nd order

② $(x'''(t))^2 + x(t) = 0$ 3rd order

comment The $x(t)$ is always the unknown & there may be other known functions like $f(t)$.

Linear versus non linear

$$x''(t)\,a(t) + b(t)\,x'(t) + c(t)\,x(t) = f(t)$$

Given   given

$x(t)$ unknown.

This is a linear second order NON-homogenous eq.

If $f(t) \equiv 0$ then its linear second order Homogenous eq.

$$(x''(t))^2 a(t) + b(t)\,x(t) = f(t)$$

is NON-linear second order ode.

First order linear eq.

consider

$$a(t)\, y'(t) + b(t)\, y(t) = f(t)$$

re-Arrange to have

$$y'(t) + b(t)\, y(t) = f(t)$$

of course b & f are redefined now.

Method of integrating factor.

If it was $z'(t) = f(t)$ we just integrate. So we want to do a change of variables to make it like this.

consider

$$y'(t) + b(t)y(t) = f(t) \qquad (1)$$

multiply by $\mu(t)$

$$\mu y' + b y \mu = f\mu$$

and pick μ s.t.

$$\frac{d}{dt}[\mu y] \underset{\text{pick } \mu}{=} \mu y' + b y \mu = f\mu$$

need $\mu y' + \mu' y = \mu y' + b y \mu$

sufficient $\boxed{\mu' = b\mu}$

check $\mu(t) = e^{\int_{t_0}^{t} b(\tau)\,d\tau}$ works). L5

So to solve

$$\frac{d}{dt}[\mu y] = f\mu.$$

integrate

$$\mu(t)y(t) - \mu(t_0)y(t_0) = \int_{t_0}^{t} f(\tau)\mu(\tau)\,d\tau.$$

So we have a formula.

Lets solve An initial value proble

$$\begin{cases} x'(t) + b(t)\,x(t) = 0 & t \in \mathbb{R} \\ x(0) = x_0 \end{cases} \quad (2)$$

x_0 & b given.