

Math 807T - Using Math to Understand Our World Workshop 2009

Introduction

In this class we're going to look at some real life problems that people really used math to solve- problems that are important to business, science, medicine, and other sectors of society. You've probably already done a bit of this in your other classes. Can you give some examples? What questions did you answer?

In this class we will face more real-world questions such as “How can you tell the time of death of a dead body?” and “How can you keep an outbreak of an infectious disease from turning into an epidemic?” or “How can you identify children who are at risk of developing adulthood obesity?” We'll start with questions of this type and then we will think about how we might use mathematics to answer the questions. This is a little bit different from what you usually do in a math class, where you are given problems that have already been formulated as math questions, but it is very much like what is done in the real world.

These types of questions are the type that many *applied mathematicians* study. An applied mathematician is someone who describes a real-world situation using the language of mathematics and then uses that mathematics to answer important questions- basically applying mathematics to the real world. A *pure mathematician* is someone who proves theorems about mathematics. Often applied mathematicians use pure mathematics, for example, a lot of pure mathematics went into developing RSA cryptography. Remember all those theorems! This semester we will be applied mathematicians.

One note: *Applied mathematics* does not mean *everyday mathematics*. Everyday mathematics is the type we use to answer questions like: “If carpeting cost \$7.50 per square foot, how much will it cost to carpet a 9×16 ft room?” or “If *Limited Too* is having a 40% off sale, how much is that \$56 dress that I’ve been wanting?”. Everyday mathematics is important, but it is not quite the same as applied mathematics. In everyday mathematics, it is clear what the math question is. In applied mathematics, while it might be clear what the question is, it is not always clear what the *math* question is (for example, you know that you want to know if a teacher is cheating on a standardized test, but you may not know how exactly to use mathematics to determine that). Applied mathematicians must think first about how to put the problem into mathematical language. Even before that, they must understand the problem well, be it in biology, finance, medicine, business or whatever. And then the mathematical analysis used to answer the question is usually more complex, with more steps, than in everyday mathematics. Many of the problems we will address in this workshop will be closer to everyday mathematics than applied mathematics because it takes more than a day to solve an applied mathematics problem. But the problems in this workshop have been chosen to develop your ability to deal with more complex applied mathematics problems which you will face in the projects.

Coming up with a mathematical statement of a real-life problem is called *modeling*. Sometimes a model can be just a simple equation. Here’s a popular example. It turns out that crickets chirp faster when it’s warm outside and slower when it’s colder. In fact, you can estimate the temperature by counting the number of times a cricket chirps in a minute. If C is the number of chirps in a minute, then the temperature T is $T = C/4 + 37$. The equation $T = C/4 + 37$ is a *model* of how the temperature depends on the number of chirps (or vice versa). It’s not exact- if you count the number of chirps, divide by 4, and add 37, most of the time you will not get exactly the current temperature. But it’s close. It’s a good model of what happens in real life. How do you think biologists came up with this model?

Solving problems in applied mathematics can be an arduous endeavor. As applied mathematicians, we will have to read a lot of background information about our real-life problems. We will need to know what factors affect our real-world system and how these factors interact. We may have to try out several models before coming up with one that seems to work. Or we may use an already-existing model, but we may have to read a lot in order to understand how to use it. Solving a real applied mathematics problem is so involved that

we will only solve 5 or 6 problems this semester, once we complete the workshop.

You've probably got the idea by now, that an important part of applied mathematics is coming up with the models. Often models are written in terms of functions that show how one quantity depends on another quantity (for example, temperature and chirps). For this reason, we will first spend some time reviewing what you already know about functions and learning about some new functions. The most common types of functions used in modeling are: linear, polynomial (usually quadratic or cubic), exponential and trigonometric.

Today we will discuss linear functions, polynomials, and exponential functions, but mostly exponential functions.

1 Linear Functions

What do you know about linear functions? What makes a function linear? What would the graph of a linear function look like? A table? A formula?

Is the cricket function linear or not linear? How do you know?

Is this linear? How can you tell?

0	0
.1	.01
.2	.04
1	1

If you're given some data from an experiment, how can you tell whether or not it is linear? Here is some data about the temperature of a cooling cup of coffee. Is it linear?

Time (in mins)	Temperature ($^{\circ}$ F)
0	200
10	181
20	163
30	146

One thing to keep in mind is that real life data is usually not as pretty as made-up data. It doesn't always fit a linear, polynomial or exponential pattern exactly and you are stuck deciding which sort of model would be best for your problem. This data looks like it might be fairly closely approximated by a linear model, but we will see later that an exponential model is better.

Here are some exercises to limber up your linear skills, and maybe even get you into the modeling mood.

Exercises

1. Graph the two lines $y = 4 - 2x$ and $4y = 12x + 7$ (preferably without a calculator). Where do they intersect? Be precise.

2. *Acme Car Rental* offers cars at \$40 a day and 15 cents a mile. *Zoomy Car Rental* offers cars at \$50 a day and 10 cents a mile. For each rental company, express the rental cost mathematically if you are going to rent the car for three days. Which company offers the better deal?
3. Consider the problem of search and rescue teams trying to find lost hikers in remote areas of the West. To search for an individual, members of the search team separate and walk parallel to one another through the area to be searched. If the search team members are close together they will be more likely to be successful than if they are far apart. Let d be the distance between searchers (and suppose they are all the same distance apart). In a study called *An Experimental Analysis of Grid Sweep Searching*, a lot of data about searcher distances and success rates was recorded. The following table comes from that report

Separation distance	Percent found
20	90
40	80
60	70
80	60
100	50

Write a function $P(d)$ to model the success rate. If $d = 0$, what is P ? Does this make sense? If $P = 0$, what is d ? Does this make sense?

4. Consider the models you used in the previous questions. In which cases do you think your model was exact and in which cases do you think it was a good approximation? Support your answers.
5. You can often use linear functions (and other types of functions) to represent a trade-off between two things. For example, the function in 3 might be thought of as representing a trade-off between searcher distance and success rate. Here's another trade-off that one of your students might face. Emma goes into the candy store to buy some tootsie rolls and some York peppermint patties. Tootsie roles are 2 cents each and peppermint patties are 5 cents each. Emma has \$1 and she's going to spend it all. She needs to figure out how many of each to buy. What are her options? Write a linear function to represent her options and draw a graph that shows all the options. Label your axes. If Emma buys 14 peppermint patties, how many Tootsie rolls can she buy? What if she buys only 6 peppermint patties? (Remark: it's ok that your graph includes fractional values of candies- we'll just ignore those values for this application).

6. Governments must make trade-offs similar to the one that Emma had to make in Exercise 5. For example, there is an ongoing battle in the government as to how much money to spend on defense programs and how much to spend on social programs. This is the famous “guns and butter” problem. There is only so much money and the more you spend on defense, the less you have for butter and vice versa. Suppose the government has \$12,000,000 to spend on guns and butter and guns cost \$400 each and butter costs \$2000 a ton. Write a linear function that represents the governments options and draw a graph. Label your axes.
7. Here is a more complicated trade-off problem. For now, just read this problem. It will be our fourth project later this semester. One day, a few years ago (in the days of VCR’s), Wendy Hines received the following somewhat desperate email from her friend Tom:

Dear Wendy,

VCR tape will record 120 minutes in LP mode: it will record (3X) 360 minutes in EP mode. We frequently record tapes for later viewing. If a movie is e.g. 137 minutes, it obviously will not fit on the tape in LP mode (17 minutes short). If I want to record most of the movie in the best quality mode (i.e. LP=60 min.of recording per hour)then I must record some portion in the slower EP mode(120 min per hour) in order to get most of the movie recorded in the better quality LP (60 min per hr) mode. With my inadequate and antiquated memory of math, I often guess at the necessary mix of recording speeds.. Unfortunately, I sometimes estimate wrongly-resulting in missing the last few minutes of a movie....very frustrating!!! Intuitively, I know there must be an algebraic formula to indicate how much EP and LP recording time must be allocated....but I cannot come up with a successful formula.....Your challenge: is there such a formula that can be relied upon that is better than my “guessing” and prevent “short” taping incidences. If there is such a formula..it could save my marriage.

Thanks for listening,

Tom

Here’s where the work of an applied mathematician really starts. Tom, who is a psychologist, has not stated his problem in a way that is very easy to understand, and he has stated it in English, not math. Our main challenge when we get to this project will be to figure out exactly what Tom is saying and figure out exactly what his trade-off is.

2 How Does Your Function Grow?

Linear functions grow at a constant rate- i.e. they grow (or decrease) by the same amount from step to step. But in real life there are a lot of things whose growth rate is always increasing or decreasing. Imagine, for example, a glass of cold lemonade warming up in a hot room. Does the lemonade change temperature by the same amount every ten minutes? How do you think a graph of the temperature might look?

Suppose a population of bacteria doubles every 6 hours (which it is likely to do). Does it increase by the same amount every 6 hours?

To know what functions to use for a model, we have to have some understanding of how different functions grow or decrease. In this section we'll explore this some.

Task 1: *Use your calculator to graph the functions $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4$ and $f(x) = x^5$ for x between 0 and 10 and draw graphs on the same axis here. How do these functions compare?*

These, of course, are examples of polynomial functions, but a polynomial function can have many terms. For example $3x^3 + 2x^2 - 6x$ is a polynomial. If the highest power in your polynomial is 2, then we might call the polynomial a *quadratic function* (though it's still a polynomial, too). If the highest power is 3, we call the polynomial a *cubic function*. What do we call it if the highest power is 1?

People who do modeling as a profession need to know a lot about how different types of functions behave, how to make a function have certain values at certain places, and grow or decrease in the right way at other places. It takes a lot of training to become fluent at modeling.

Task 2: *Graph the polynomial function $f(x) = 3x^3 + 2x^2 - 6x$. How low does it go? How high does it go? Where is the function zero?*

Usually an applied mathematician has to work backwards. She or he has to take a graph (probably made up of data points) and find a function that nearly fits it.

In the next few tasks we will learn something interesting about polynomial functions.

Task 3: *Make a table that shows the values of the function $f(x) = 2x$ for integer values of x (use $x = 0, 1, 2, 3, 4, 5, 6$). Add an extra column onto your table and in that column write the differences between consecutive entries in the second column. Now make another table for $f(x) = x^2$, only this time add on two extra columns and in each of those columns write the difference between consecutive entries in the previous column. Do the same thing for $f(x) = x^3$ and $f(x) = x^4$. What do you notice?*

Task 4: *Try the same thing for $f(x) = x^2 - x$ and the function in Task 2. What happens?*

Task 5: *How might what you just learned be useful? Suppose a laboratory scientist is studying how much a metal rod will stretch when you pull on the ends. He does some experiments and takes some data which is in the table below. What sort of function should you use to model the amount of stretching as a function of the applied force (you don't have to come up with the specific function, just figure out what sort of function you would want to look for)?*

Applied force	Amount of stretching
1 N	.09 mm
2 N	.76 mm
3 N	2.61 mm
4 N	6.24 mm
5 N	12.25 mm
6 N	21.24 mm
7 N	33.81 mm

In real life, data is never this nice. The differences never work out exactly. You might get the following data and then when you work out the differences, you would have to decide if the 3rd differences were close enough to being constant. Once you decided to go with a cubic polynomial for your model, you'd have to then figure out exactly which cubic polynomial works best. There's a lot of work, and math, involved in coming up with polynomial models. But once you finally have your model, you can use it to predict how much the rod will stretch under even greater forces, or just under forces that you haven't tried yet (like 4.5 N). This ability to predict is crucial in science and industry.

Applied force	Amount of stretching
1 N	.08 mm
2 N	.76 mm
3 N	2.6 mm
4 N	6.25 mm
5 N	12.27 mm
6 N	21.24 mm
7 N	33.8 mm

By the way, the cubic polynomial that fits the data (exactly!) in the first table is $s(f) = .1s^3 - .01s^2$. The data in the second table comes pretty close to matching this.

Task 6: *Bacteria often reproduce by simply splitting in two, and then each half grows to the size of the original one. Imagine the following scenario: a single bacterium is sitting in a Petri dish filled with agar (yummy stuff that bacteria like to eat). The bacterium splits into two. Each of those grow and split into two more, so now there are four. Each of those*

four split into two more, etc. Count the bacteria after each division. Make a table that shows on one side the number of times you have counted the bacteria so far, and on the other side the number of bacteria you counted each time (let your first entry be 1 and 1). Look at the differences. What happens? Can you think up a function for the number of bacteria of the form $B = f(n)$ where n is the number of times you have counted and B is the number of bacteria? Even though this function for B matches the story exactly, it is really an approximation. In real-life, the number of bacteria is probably not exactly B , as a few bacteria might die, or a bacteria might occasionally split into three and not two, or not split at all.

A function of the form of B is called an *exponential function* (can you guess why?). Exponential functions grow faster than any power function (and hence any polynomial). Here are some graphs that show the functions $f(x) = x^2$, $f(x) = x^5$ and $f(x) = 2^x$. The power function may be bigger at first, but the exponential function *always* beats it out in the end.

Exercises (These will be handed in)

1. Below is some data showing the stopping distance of an Alpha Romeo sports car for different speeds.

Speed (in mph)	Stopping distance (in feet)
70	177
40	57.8
130	610.5
100	361
140	708
160	925

Find a model (i.e., an equation) for the stopping distance and use it to predict the stopping distance if the car were going 200 mph. (hint: assume your model has the form $y = kx^n$ where k is a fixed number and n is a power).

2. Notice that in the data above, the speeds are given every 30 mph, and the differences work out almost exactly. In real life, data is seldom so nice. Here is some more realistic stopping distance data for Toyota's new Escargo.

Speed (in mph)	Stopping distance (in feet)
35	74
50	149
90	490.5
100	590

Can you find a reasonable model for this data? This is pretty challenging, but remember that from the above exercise you have some idea of what form your model might take. Once you've got a model you're happy with, use it to figure out the stopping distance at interstate speeds (assuming you're going the speed limit).

3. Gulliver, in his travels, discovered that the Lilliputians were increasing 2.6% in population each year. Here is a table of the population for the ten years that Gulliver spent in Lilliputia (the population is measured in thousands).

Year	Population (in thousands)
1780	255.97
1781	262.63
1782	269.46
1783	276.46
1784	283.65
1785	291.03
1786	298.59
1787	306.36
1788	314.32
1789	322.49

If Gulliver goes back to Lilliputia in ten years (i.e., in 1799) how many Lilliputians will there be? What will the increase have been between 1798 and 1799, both in raw numbers and in percentages? What if he goes back in 40 years? In 50? What's going to happen to the Lilliputian population in the long run? How does this compare to the bacteria population we talked about earlier?

- Graph the functions $f(x) = x^{20}$ and $f(x) = 1.5^x$. Which function is bigger? Explain.
- In which problems did you use exponential models and in which did you use power models?

3 More About Exponential Functions

Modeling with polynomial functions is a very interesting and deep topic that we have only scratched the surface of. Unfortunately we don't have time to do more than that. But at least this gives you a little flavor for the topic.

We will spend the rest of the Workshop talking about exponential functions. They are very useful in modeling- they come up all the time, more often than polynomial functions -and they are not too hard to work with. Here is an everyday example where exponential functions describe what's happening.

Task 1: *Suppose you deposit \$100 into a savings account and the savings account earns 8% interest. Normally interest is compounded (i.e., added on) several times a year. Let's suppose that in this case interest is compounded quarterly (i.e., every three months). That means that $8/4\%$ is added on each quarter (you divide the interest rate by the number of times it is compounded each year). So after the first three months the bank adds \$2 to your account. How much money will you have after 2 quarters if you always put the interest back into your savings account? 3 quarters? a year? Can you find a function, $M(q)$, that tells you how much money you will have after q quarters?*

Remark: Normally when a bank lists interest rates for savings accounts, they list two numbers- the regular interest rate (or *nominal* rate) and the APY. APY stands for *annual percentage yield* and is the percentage increase of the principal in a year's time. Normally, unless the interest is compounded only annually, the APY is a bit larger than the nominal rate.

Task 2: *What is the APY for the 8% savings account above? When Wendy Hines wrote these notes, she checked out Wells Fargo and found out that they give a whopping 2.23% interest rate for accounts between \$10,000 and \$25,000, compounded quarterly. What is the APY?*

Task 3: *We've used exponential functions to model the value of a bank account accruing interest and the size of a growing population. How are these two problems similar?*

Before, we said that exponential functions grow faster than almost every other function, but it may seem that at a 2.23% interest rate, or even at an 8% interest rate, your money doesn't grow too quickly. Sometimes you have to wait a bit for the exponential function to really take off. Here's an old story about a forgotten savings account- it goes something like this. One day, in 1996, a man by the name of Samuel Johnson was in his attic going through a trunk that had belonged to his grandfather. In the trunk he found an old bank passbook that had apparently been in his family for a long time. The last transaction in the passbook was dated July 31, 1790. The balance at that time was \$244.82. On top of the page was printed the current interest rate: 4.5% compounded quarterly.

Task 4: *How much money was the account worth in 1996?* Samuel took the passbook to the bank, which still existed, but they no longer had any record of the account and they weaseled out of paying Samuel his money.

Exponential functions may grow slowly at first, but as you see, at some point they will really take off. Any time you put some money in the bank and don't touch it and let it accrue interest (i.e., let the interest compound), the value of your savings will be given by an exponential function.

Compound interest has a huge impact on how debts and savings grow. If you google "compound interest" you will get literally millions of hits, most of them from financial companies trying to explain to you how compound interest makes you a lot of money, and why, to get the full effect of compound interest, you need to start saving money early.

Task 5: *For example, suppose at age 40, you invested \$5000 in a money market fund that makes 6% compounded annually. How much will this investment be worth when you retire? What if you had made the investment when you were 25? What if instead your parents had invested it for you when you were born? What if they had been able to find a fund that paid 9% annually?*

If you added \$100 a month to the fund, your money would grow even faster. You'd be amazed! It takes some more math to calculate savings when you're also making a monthly or yearly contribution, so we'll save that for a project later in the semester. In that project, we will also see how credit card debt grows exponentially, if you don't pay it all off each month.

One way that people like to think about exponential functions is to talk about doubling times.

Task 6: *If you invest \$100 at an interest rate of 8% compounded quarterly, how long does it take for your investment to double (assuming that all interest is put back into the account and that you don't add or take out anything from the account)? How long does it take for it to double again? And again?*

Each exponential function has its own fixed doubling time. Exponential functions grow so quickly because more and more is doubled each time.

Task 7: *What is the doubling time for the \$5000 investment at 6% compounded annually? When will it double again? How much money will you have after the first doubling time? After it doubles again? And again? What is the doubling time for the function $f(x) = 1.5^x$?*

Task 8: *There is a famous story about the meeting between a Chinese emperor and the inventor of the game of chess. The emperor was so delighted by the new game that he offered the inventor anything he wanted in the kingdom. The inventor said that all he wanted was some grains of rice. “I would like one grain of rice on the first square of the chessboard, two grains on the second, four grains on the third, and so on. I would like all of the grains of rice that are put on the chessboard in this way.” Thinking this would amount to no more than a bushel of rice, the emperor readily agreed. Let’s try this. What do you think will happen? How many grains of rice will be on the last square? (Extra credit if you can figure out how many were on the entire chessboard). Do you think this is more or less than a bushel? Notice that the amount in the last square is more than the amount on all the previous squares put together. In fact, this is true for every square: the amount on any square is more than the amount on all the previous squares put together. This shows how doubling can make things grow amazingly quickly.*

Task 9: *Discuss: which would you rather (1) I give you \$10000 a year for 64 years or (2) I give you \$.01 the first year, \$.02 the second year, \$.04 the third year and so on, doubling the previous years amount each year, for 64 years?*

The enormity of exponential growth has very important real-world implications. With these examples in mind, think about what it means to say something like “world oil consumption is growing at a rate of 2.3% per year”. In 2000, world oil consumption was about 27,740,000,000 barrels. The total amount of oil believed to remain in the earth is about 1027 billion barrels. If we do the math and add up the total amount used for the next several years (finding out how much is used in 2020 is not hard, but adding up the total amount used between now and 2020 is a little harder), we would discover that we will run out of oil in about 27 years, unless consumption is drastically reduced. Even still, there is a finite amount of oil in the earth, and we will run out sooner or later.

Task 10: *If oil consumption continues to grow at a rate of 2.3% per year, how long until consumption doubles? How many barrels will be used per year then? How many barrels will be used per year after it doubles a second time?*

When we do our project about exponential growth of credit card debt, we will be able

to apply the math we learn there to predict growth of the national debt, which is currently a little over \$8 trillion. Scary, huh? And we will be able to show that the world will run out of oil in 27 years.

Every exponential function has the form $f(x) = Cb^x$ where C and b are fixed numbers. C is called the *initial value* because it is the value of $f(x) = Cb^x$ when $x = 0$, for example in the interest problems, C is the initial principal. What is C in the rice problem? The number b is called the *base*. It can be any positive number. What is b in the interest problems? in the bacteria problem? in the rice problem?

Exercises (These will be handed in)

1. Review the uses of exponential functions we have talked about so far. In which examples are exponential models exact and in which are they approximations?
2. Using your results from Exercise 3 in Section 2, what is the doubling time for the population of Lilliputia? The world's population currently has a doubling time of about 38 years. How big of a problem do you think this is?
3. What are C and b for the Lilliputian population model in Exercise 3 of Section 2?
4. If a 5% interest rate is compounded monthly, what is the APY?
5. Suppose Wendy Hines initially invests \$1000 in the account from Exercise 4 when her daughter is born. Write a function that shows how much the account is worth after n years. Sketch a graph of this function that goes up to 90 years. How much will the account be worth when she goes to college? When she retires?

4 The Exponential Number e

The most common base for exponential functions is the number e . The value of e is about 2.718. Does your calculator have an e button? If so, you can type e^1 and see several digits of e . To write down the digits of e exactly would require an infinite number of digits after the decimal place, and so it is easier to just write “ e ”. You might wonder why such a weird looking number is so popular. Unfortunately that’s pretty hard to explain. One early appearance of e actually came out of the work of a mathematician named Jacob Bernoulli, in the late 1600’s (there was a whole family, including three generations, of Bernoulli mathematicians). Bernoulli wanted to understand how compound interest worked to cause investments and debts to grow. He came up with the formula $P(1 + r/n)^n$ to describe the value of an investment with principal P invested at a rate r compounded n times per year.

Task 1: *When we say “value” what do we mean? the value when? Is this formula the same as the one you found for compound interest?*

Bernoulli wondered, “What happens if I compound more often? Will that have a big effect on how fast the value of an investment increases?” So he compared, for example, the growth of investments with quarterly compounding, monthly compounding and daily compounding.

Task 2: *Does it make a big difference how often interest is compounded?*

Bernoulli, being a mathematician, wondered what would happen if compounding was continuous. What does it even mean to “compound continuously”? Well it’s more often than every hour.

Task 3a: *What would n be if you compounded every hour?*

It’s more often than every minute.

Task 3b: *What would n be if you compounded every minute?*

It’s more often than every second.

Task 3c: *What would n be if you compounded every second?*

Suppose, for the moment, that the interest rate is $r = 1$ (in percents that’s 100%- a good deal!) and that your initial investment is \$1.

Task 4a: *What would the balance be after a year if you compound hourly? every minute? every second? What happens to the expression $(1 + 1/n)^n$ as n gets larger and larger?*

We say that

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n .$$

It turns out that

$$e^r = \lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n .$$

Task 4b: Use your calculator and different values of r to convince yourself of this. Record the computations that you did here.

If we compound *continuously* then after a year, with a principal of P and an interest rate of r , we have

$$P \lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = Pe^r$$

dollars.

Task 5: Give a formula for the value of the investment after ten years. After t years.

The number e is not a very good base for investment formulas that don't use continuous compounding, nor is it very good for the bacteria division or rice problem, but it turns out to be very convenient for many scientific applications.

Exercises

1. Graph e^x and e^{2x} . How do they compare? Now graph $e^x \cdot e^x$ and e^{2x} . Make an observation. Do you remember any laws of exponents that could account for this observation?
2. Suppose you are lucky enough to find a bank with an 8% interest rate compounded continuously. Suppose you deposit \$100. Write a formula for the amount of money in the account after t years. What is the APY?

5 The Undoing of the Exponential Function

Task 1: *On your calculator, compute e^2 . There is another curious button on your calculator called \ln . Compute \ln of the number you just got. What happened? Now compute e^{10} and then compute \ln of that number. Compute $\ln(e^{3.4})$. Compute $\ln(e^x)$ for a few other values of x , or even graph $\ln(e^x)$. What happens? There's a word for this. We say $\ln x$ is the _____ of the exponential function e^x .*

VERY IMPORTANT FACT: $\ln(e^x) = x$

Task 2: *Suppose we invest \$100 at 8% compounded continuously. Can you use the **VERY IMPORTANT FACT**, instead of trial and error, to figure out how long it takes the investment to double? In order to find out how long it takes the value of the investment to double, do you really need to know the amount of the principal? Why or why not? How long would it take the investment to double if the investment rate were 7%?*

Task 3: Solve the following: $4e^{3x} = 24$, $400e^{0.1t} = 1000$

Every exponential function has its own associated logarithm function. The function $f(x) = 2^x$ has the associated logarithm function $g(y) = \log_2 y$. The function $f(x) = 10^x$ has the associated logarithm function $g(y) = \log_{10} y$. The function $f(x) = b^x$ has the associated logarithm function $g(y) = \log_b y$. The number b is called the *base* just as it is for exponential functions. The natural logarithm is really just \log_e . It's called the *natural logarithm* just because it's used so often.

Task 4: *What is the base of $\log_2 y$? $\log_e y$? $\ln y$?*

Task 5: *Write a VERY IMPORTANT FACT for \log_2 . What is $\log_2 8$? $\log_2 32$?*

In this class, we'll only use the natural logarithm. The algebra of exponentials and logarithms is very useful in applied mathematics and is a topic in Algebra II. Unfortunately, we don't have time to study this topic in any depth. We will just use logarithms to help us solve problems.

Exercises

1. Suppose we deposit \$5000 into a bank account that gives 5% interest compounded continuously. Use \ln to determine when the balance in the account would be \$1,000,000.
2. Repeat Task 5 for $\log_{10} y$. What is the base of $\log_{10} y$? What is $\log_{10} 10$? $\log_{10} 10000$? $\log_{10} 3$? $\log_{10} 30$?
3. What happens if you take the \log_b of a negative number (for any base b)?
4. Graph $\ln x$. Explain why the graph looks the way it does (the next problem might help you with this one).
5. Graph both $\ln x$ and e^x on the same graph. Do you notice any graphical relationship between these two functions? Can you think of a reason why they might be related in this way? You could also compare $\log_{10} x$ and 10^x if you want another example to look at.

6 Review

Is your head spinning? Let's review. We talked a little bit about linear functions. We talked even less about polynomials. We saw a few situations in the exercises that we could model using linear functions and we saw an example where we could use a quadratic model (the stopping distance exercise). We talked a lot about exponential functions. Linear functions have a steady increase, polynomials grow faster, but exponentials grow the fastest. Many real-life things grow exponentially. We talked a lot about how investments grow exponentially if you leave them sit and always add the interest back in. We also saw that bacteria division and rice-grain doubling are exponential. We talked about examples of exponential population growth. If something increases by a fixed percent over each time period, then it is growing exponentially.

We talked about the special exponential base e , which we will use quite often in the weeks to come. We also saw how logarithms “undo” exponentials and how to use \ln to solve equations with e . We learned how to find the doubling time of exponential functions. These skills will soon be very useful as we try to solve some real-world problems.

Review Exercises (These are to be handed in)

Here are some problems to try that will help you to consolidate what we have learned today and help you move your focus towards exponential functions with base e .

1. Describe some things you might do to determine whether some given data is linear, polynomial or exponential. You may refer to exercises or tasks.
2. Populations are often thought to grow exponentially until they begin to run out of resources, at which time they begin to level off. Suppose we have a population of rabbits in a park and on May 1 we did a rabbit census (don't laugh- people really do this) and found that there were 426 rabbits in the park. If t stands for the number of days since May 1, then we might model this population as $R(t) = 426e^{rt}$ where r is the per capita growth rate per day (that is, the number of new rabbits each existing rabbit makes, on average, per day). Presumably (hopefully!) r will be substantially less than one. How do we measure r in practice and how do we use this model to predict things about the growth of the population? Well suppose our park workers do a second census on June 1 and find that there are now 600 rabbits. How could you use that information to find r ? Find r and use that to determine the doubling time for the rabbit population. What will the population be on Sept. 1? Why is this information useful? Notice that with this model, we could have fractions of rabbits. That's ok, it's just an approximation.

3. Brazil experienced exponential inflation during the last half of the 20th century (before its currency was revalued). Suppose the price of a loaf of bread was given by $b(t) = .35e^{.34t}$ where t is years since 1950. What was the price of bread in 1950? (the Brazilian unit of currency is the real- pronounced “hay-yal”)? What was the price in 1995? You can probably imagine why they got rid of lower denominational notes. By 1995, the smallest note was the million real. What was the yearly inflation rate during this period? By what year was the price of bread one real?
4. Solve
- (a) $3e^{2x} = 20$
 - (b) $170e^{.7x} = 420$
 - (c) $\ln(e^x) = 1$ (this one is really quick if you spot the trick)
5. Fill in the blanks. If oil consumption increases by a fixed percentage each year (i.e., the same percentage every year) then we say that consumption is growing _____. If a savings account grows with a fixed interest rate year after year (and you don’t take any money out or put any money in, except for the interest) then your savings grows _____. If a population doubles every 10 years then it is growing _____.
6. If we deposit \$1000 and are lucky enough to find a bank that compounds continuously, write a function that describes the value of our savings after t years if the interest rate is 7%. Assume that we don’t withdraw or deposit any money into the account after the initial deposit, except that we always return the interest back to the account.
7. Suppose some quantity can be modeled as $q(t) = q_0e^{rt}$. There is a rule called *The Rule of 70* which says that the doubling time is approximately $70/r$. Where does this come from? How accurate is it?

7 Exponential Decay

In most of our previous examples, the exponent was always positive, but it can be negative, too.

Task 1: Graph the function $f(x) = e^{-x}$. What is $f(0)$? $f(1)$? $f(3.5)$?

We say that such a function *decays exponentially*.

Here is another example of exponential decay. Consider a full glass of water in a straight up and down glass. Now pour half of the water out and note the new height of the water. Now pour half of that water out and note the height again. Continue doing this.

Task 2: Write a function $h(n)$ for the height of the water after the n th pouring (so $h(0) = 1$). Notice that here the base is less than 1. How is this like having a negative exponent?

Here is some space to draw the function $f(x) = (1/2)^x$ (in the example above, n could only be an integer, but x can be anything).

There are many real-life examples of exponential decay. Perhaps the most famous one is radioactive decay. All atoms are made up of protons, neutrons and electrons. Uranium atoms have lots and lots of protons, neutrons and electrons and, because uranium atoms are very unstable, sometimes these particles go zinging off to other places (we say the uranium “decays” or “breaks down”). This is called radioactivity. The particles that zing off are called *alpha* particles and *beta* particles and they can do damage to living cells that they zing into. Once alpha particles and beta particles have zung off, the remaining atom is no longer a uranium atom. The new atom is also radioactive, however, and more alpha and beta particles will zing off. Eventually, as alpha and beta particles zing off, the uranium will be transformed into something no longer radioactive, but this takes a long time.

Uranium is a well-known radioactive element, but because it decays into other radioactive elements, it can be a little confusing to talk about the decay of uranium. Instead, lets talk about the decay of radioactive iodine isotopes (an isotope is a version of an element that has a different number of neutrons than the regular version does- isotopes tend to be radioactive). Iodine-129 and iodine-131 are both radioactive, but when they undergo radioactive decay, they turn into nonradioactive elements. As time goes on, more and more of the iodine isotope decays until eventually there is no more radioactive iodine left. The remaining substance is safe. It turns out that if I_0 is the amount of iodine isotope initially in a lump of stuff (measured in milligrams perhaps), then $I(t) = I_0e^{-rt}$ is the amount of iodine isotope after time t where the decay rate is r . This is really an approximation; it won't be exact, but will be pretty close. Decay rates for most radioactive substances have been determined in the lab.

Remark: \$1 for anyone who can find the contradiction in the above paragraph.

What chemists usually measure, rather than the decay rate itself, is what's called the *half-life*. This is analogous to the doubling time in exponential growth. The half-life is the time it takes for half of the radioactive substance to decay.

Task 3: *What would r be if the half-life of a radioactive substance is 20 years? (be careful with the signs)*

As mentioned, when uranium undergoes radioactive decay, other radioactive elements result. The half-life of uranium is about 760 million years, but in nuclear reactors uranium

is broken down much more quickly than that. Two of the byproducts of the decay of uranium are radioactive iodine-129 and iodine-131. These are part of what we call “nuclear waste”. The half-life of iodine-129 is 15.7 million years and the half-life of iodine-131 is 8 days. Here are two problems about iodine-129 and iodine-131.

Exercises (These are to be handed in)

1. The Snake River Plain aquifer is the most important underground water resource in the northwest U.S. It is the sole source of drinking water for 200,000 people. It is the main source of irrigation water for crops and fisheries in Idaho. Over 75% of trout eaten in the U.S. comes from Idaho fisheries (and where do your potatoes come from?). In 2001, a PhD student named Michelle Boyd at the Idaho National Engineering and Environmental Laboratory (INEEL) measured the amounts of various nuclear contaminants in the aquifer. She knew that there were problems because INEEL used to be a big producer of nuclear weapons from 1950 until the end of the Cold War, and they took little care to dispose of their nuclear waste carefully. Basically they just dumped it into the aquifer. Nuclear waste is no longer being dumped into the aquifer, but of course what’s already been dumped is still there. She found out that there are areas in the aquifer where the amount of iodine-129 is 3.82 picocuries per liter of water (a curie is a standard unit of radiation- it would be hard to explain exactly what it means; a picocurie is a trillionth of a curie). The highest amount that is considered safe by the FDA is 1 picocurie per liter. How long will it be until water in the aquifer is safe?
2. I-131 is sometimes used in medical imaging. It is injected into the blood and will collect on certain kinds of tumors. It is also used to treat hyperthyroid (overactive thyroid). When administered to a patient, I-131 (because it’s iodine) accumulates in the thyroid where it decays. As it decays, the particles that zing off kill part of the gland, which is good if your thyroid is overactive. I-131 has a half life of 8 days. Suppose it takes 72 hours to ship I-131 from the producer to the hospital. What percentage of the original amount shipped actually arrives at the hospital? Suppose it is stored at the hospital for another 48 hours before it is used. What percentage of the original amount is left when it is used? How long will it be before the I-131 is completely gone?