Take-Home Test 5

Due: Thursday, November 12

Collaboration Allowed

Collaboration on this test is both allowed and encouraged. By collaboration, I mean that you are encouraged to discuss the problems, test out your ideas, check your reasoning and arguments, etc., with other people. *However, there is a big difference between collaborating and copying. Each student must write up his or her own solutions to the problems in his or her own words.*

As before, you will be graded both on mathematical content and on clarity of expression. Exercises 1-3 are worth a maximum of 12 points each and Exercise 4 is worth a maximum of 24 points. Some of the questions are a bit open-ended, so be creative, make educated guesses if you have to, but back up your assertions by providing proofs, counter-examples, or (in the case of guesses) numerical evidence. In writing your answers, use complete sentences (with punctuation!) and be sure to say exactly what you mean. Papers will be graded on the basis of what you have written, so be sure to take the time to express yourself clearly. If you are stuck on a problem and have no idea where to begin, a good way to get started is to look at lots of specific examples and try to find a pattern.

Exercises:

(1) In this exercise we investigate the *perfect 3-pile in-shuffle.* Take a standard deck of 52 cards and throw out one card. Cut the 51-card deck into three equal piles of 17 cards each: the top pile, the middle pile, and the bottom pile. Now take the bottom card of the top pile and put it on the bottom of a new pile. Next, take the bottom card of the middle pile and put it on top of the card you just laid down. Then take the bottom card of the bottom pile and put it on top of the card from the middle pile. Repeat this process until all cards have been added to the new pile.

   (a) Suppose a card starts out x positions from the bottom of the original un-cut deck (so $x = 1$ means the card is at the bottom of the original deck). How many positions from the bottom does it end up in the new deck after one perfect 3-pile in-shuffle? How about after two perfect 3-pile in-shuffles? k perfect 3-pile in-shuffles? For each question, you should end up with one formula which gives the answer and which will most likely involve a congruence. Be sure to explain how you got your formula.

   (b) Is there a positive integer k such that after k perfect 3-pile in-shuffles the deck is returned back to its original order? If so, what is the smallest such integer? (Be sure to show your work here!)

(2) Fun with the Euler Phi Function.

   (a) Make a list of the values of $\phi(n)$ for $1 \leq n \leq 50$.

   (b) Prove that $\phi(n)$ is even for any integer $n > 2$. (*Hint:* Consider two cases: when n equals a power of 2 and when n does not equal a power of 2.)

(3) Recall *Euler's Theorem:* Let a and m be integers such that $m \geq 1$ and $\gcd(a, m) = 1$. Then $a^{\phi(m)} \equiv 1 \pmod{m}$.

   (a) Show that Euler's Theorem can be false if $\gcd(a, m) \neq 1$ by providing a specific counter-example.

(b) Euler's Theorem is helpful in computing powers modulo composite numbers. Use Euler's Theorem to find $2^{9999950} \pmod{441}$.

(4) For a positive integer n, we define $n!$ to be the product $(n)(n-1)(n-2)\cdots(2)(1)$. For example, $5! = (5)(4)(3)(2)(1) = 120$. By convention, we define $0!$ to be 1. Let n and k be two non-negative integers such that $n \geq k$. The *binomial coefficient* $\binom{n}{k}$ is defined to be the number

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.$$

It is true that $\binom{n}{k}$ is always an integer; you do not have to prove this fact.

(a) Find $\binom{8}{2}$, $\binom{10}{4}$, $\binom{7}{5}$ and $\binom{12}{7}$.

(b) Prove that for all positive integers n and k such that $n \geq k$, we have the equality

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

(c) Let a, b, n be integers with $n \geq 1$. Use the Principle of Mathematical Induction and part (b) to prove the *Binomial Theorem*:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{i}a^{n-i}b^i + \cdots + \binom{n}{n-1}ab^{n-1} + b^n.$$

(*Hint:* It might be helpful to note that $(a+b)^n = a(a+b)^{n-1} + b(a+b)^{n-1}$.)

(d) Let p be a prime number. Is it always true that p divides $\binom{p}{k}$ when k is an integer such that $1 \leq k \leq p-1$? Either carefully explain why this is true or provide a specific counter-example showing it is not true.

(e) Let p be a prime and a, b be integers. Prove *The Freshman's Dream*:

$$(a+b)^p \equiv a^p + b^p \pmod{p}.$$

(f) Give an example to show that "The Freshman's Dream" does not necessarily hold if p is not prime.

Statement of Sources: Give a list of all people with whom you discussed the exercises on this test. Also, if you used any references besides the class notes, list them as well.