Take-Home Test 2

Due: Thursday, September 24

Solo – No Collaboration Allowed

On this test, your work is to be your own with no consultation with any other person (in this class or not) except for the instructor. Feel free to ask me any questions. I won't give you any answers to the exercises but will be happy to try to clarify any confusion you may have, probably by asking you more questions. You may feel free to use any written references or books, just not any consultation with any persons. I prefer that you do not use any internet sources, but if you can't resist, be sure to acknowledge that in the name of academic honesty.

As usual, you will be graded both on mathematical content and on clarity of expression. Each problem is worth a maximum of 12 points. Some of the questions are a bit open-ended, so be creative, make conjectures, and back up your assertions by providing proofs or counter-examples. In writing your answers, use complete sentences (with punctuation!) and be sure to say exactly what you mean. Papers will be graded on the basis of what you have written, so be sure to take the time to express yourself clearly. If you are stuck on a problem and have no idea where to begin, a good way to get started is to look at lots of specific examples and try to find a pattern.

Exercises:

(1) (a) Use the Euclidean Algorithm to find the greatest common divisor of 6603 and 5680. Use the Extended Euclidean Algorithm to express this gcd as a linear combination of 6603 and 5680.

(b) Decide whether the equation $6603x + 5680y = -426$ has any integer solutions (x, y). If it does, find a general form for all such solutions. Explain your answer.

(2) Let a, b, c be non-zero integers. We know from high school algebra that an equation of the form $ax + by = c$ is the equation of a straight line in the plane. A point in the plane (u, v) where both coordinates u and v are integers is called a *lattice point*. Note: You might want to use graph paper for this exercise.

(a) Carefully draw a graph of the line $6x + 8y = 4$ in the plane. Does the graph pass through any lattice points? If so, list at least 4 such points and show them on the graph. Explain with reasons how many lattice points the graph passes through. If the graph does not pass through any lattice points, then explain why not.

(b) Carefully draw a graph of the line $4x + 8y = 6$ in the plane. Does the graph pass through any lattice points? If so, list at least 4 such points and show them on the graph. Explain with reasons how many lattice points the graph passes through. If the graph does not pass through any lattice points, then explain why not.

(c) Describe all lattice points (if there are any) on the graph of the equation $7696x + 4144y = 1776$. You do not need to draw the graph; just tell me about the lattice points.

(3) (a) Observe that for $n = 1, 2, 3, 4$ and 5, the values of the expression $n^2 - n + 41$ are $41, 43, 47, 53$ and 61, respectively – all primes. Is this always true? In other

words, is $n^2 - n + 41$ a prime integer for every positive integer n? Either prove this is true or give a counter-example showing it is false.

(b) For any integer $k \geq 1$, let p_k be the kth prime number, i.e., $p_1 = 2, p_2 = 3, p_3 = 5$ and so on. Note that $p_1 + 1 = 2 + 1 = 3$ is prime, $(p_1)(p_2) + 1 = (2)(3) + 1 = 7$ is prime, and $(p_1)(p_2)(p_3) + 1 = (2)(3)(5) + 1 = 31$ is prime. Is it true that $(p_1)(p_2)\cdots(p_k) + 1$ is always a prime number? If so, prove it. If not, give a specific counter-example.

(4) Factoring big numbers is a very hard problem. But big numbers which have a very special form can often be completely factored into primes by using some algebraic tricks before doing trivial divisions.

(a) Factor the polynomial $x^{24} - 1$ completely. Hint: Use the well-known factorizations of $x^2 - 1$, $x^3 - 1$ and $x^3 + 1$ (look these up if you don't remember them).

(b) Use part (4a) to find the complete prime factorization of $7^{24} - 1$. Hint: You may use the fact that 409 divides $7^{24} - 1$, but you should do as much as you possibly can before you use this fact.

Note, by the way, that $7^{24} - 1 = 191,581,231,380,566,414,400$ is a twenty-one digit number which you could factor by brute force trial divisions if your calculator carries enough digits - most don't - but this would be a very tedious task.

(5) *Twin primes* are pairs of consecutive odd integers $(p, p + 2)$ which are both prime. For example, $(3, 5), (5, 7), (11, 13)$ are all pairs of twin primes. The *Twin Primes Conjecture* (still unresolved) claims that there are infinitely many pairs of twin primes. Give three more examples of twin primes in addition to what is given in this exercise.

Building on this idea, we can define a set of "triplet primes" to be three consecutive odd integers $(p, p+2, p+4)$ which are all prime. A quick scan of our list of primes less than 100 shows that there is exactly one set of triplet primes in that range: namely $(3, 5, 7)$. Are there any other triplet primes (ever)? If so, give an example. If not, prove that there are no others.

Statement of Sources: As this was a "Solo" exam, you shouldn't have talked with anyone besides the instructor about this exam. Please write the following statement on your exam, and sign your name:

I have neither given nor received any help on this exam.

Also, if you used any references besides the class notes, list them as well.