Name: ______________________________

Quiz 9 Solutions

Please write your solutions to the following exercises in the space provided. You should write legibly and fully explain your work.

Good Luck!

(1) A diagonalization of the matrix A is given in the form $P^{-1}AP = D$. List the eigenvalues of A, their algebraic and geometric multiplicities, and bases for the corresponding eigenspaces. [10 pts]

$$\begin{bmatrix} -1/4 & 3/4 & -1/4 \\ 1/8 & 1/8 & 1/8 \\ 5/8 & -3/8 & -3/8 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 2 & 0 & 2 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 1 \\ 1 & 2 & 0 \\ -1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Solution: The eigenvalues of A are the diagonal entries of D. Thus, A has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 6$.

Bases for the eigenspaces can be found by selecting the columns of P which correspond to the diagonal entries of D. So, a basis for E_{-2} is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

and a basis for E_6 is

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

We see that $\lambda_1 = -2$ has algebraic and geometric multiplicity 2, and $\lambda_2 = 6$ has algebraic and geometric multiplicity 1.

(2) Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$. Show that A and B are not similar matrices. [6 pts]

Solution: If A and B were similar, then they would have equal characteristic polynomials.

The characteristic polynomial of A is

$$\det \begin{bmatrix} (1-\lambda) & 3 \\ 2 & (2-\lambda) \end{bmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4.$$

The characteristic polynomial of B is

$$\det \begin{bmatrix} (1-\lambda) & 1 \\ 3 & (-1-\lambda) \end{bmatrix} = (1-\lambda)(-1-\lambda) - 3 = \lambda^2 - 4.$$

Since A and B have different characteristic polynomials they cannot be similar.

(3) Is the following statement true or false? Carefully justify your answer. [4 pts]

Let A be a diagonalizable $n \times n$ matrix such that each eigenvalue λ satisfies $|\lambda| < 1$. Then $\lim_{k\to\infty} A^k$ equals the $n \times n$ zero matrix.

Solution: This statement is true. Since A is diagonalizable, there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. We know that the diagonal entries of D are the eigenvalues of A. Moreover,

$$\lim_{k\to\infty} A^k = \lim_{k\to\infty} (PDP^{-1})^k = P(\lim_{k\to\infty} D^k)P^{-1}.$$

Now, D^k is a diagonal matrix for which each diagonal entry equals the kth power of an eigenvalue of A. But, as $k \to \infty$, the kth power of any eigenvalue of A approaches 0. That is,

$$\lim_{k\to\infty} D^k$$

equals the zero matrix. Therefore, $\lim_{k\to\infty} A^k$ equals the $n \times n$ zero matrix.