

Name: _____

Quiz 8 Solutions

This is a take-home quiz. You are allowed to use your class notes and text, but no other resources (including books, internet, or people). This is due in class on Tuesday, October 28. No late submissions will be accepted.

Please write up your solutions to the following exercises. You should write legibly and fully explain your work. Staple your pages together with this page as the cover – remember to write your full name at the top.

Exercises:

- Section 4.4: # 10, 11, 16
- Section 4.6: # 8

Solutions:*Section 4.4:*

- (10) Since A is a triangular matrix, its eigenvalues are the diagonal entries. Thus, $\lambda = 3$ is the only eigenvalue of A and it has algebraic multiplicity 3. We now find a basis for the eigenspace E_3 . To do this we find the null space of $A - 3I$.

$$[A - 3I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Thus,

$$E_3 = \text{span} \left(\left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \right).$$

We conclude that E_3 has basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Since $\dim(E_3) = 1$, the eigenvalue $\lambda = 3$ has geometric multiplicity 1. By Theorem 4.27, the matrix A is not diagonalizable (the algebraic and geometric multiplicities of $\lambda = 3$ are not equal).

(11) The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} (1 - \lambda) & 0 & 1 \\ 0 & (1 - \lambda) & 1 \\ 1 & 1 & (-\lambda) \end{bmatrix} \\ &= (1 - \lambda) \det \begin{bmatrix} (1 - \lambda) & 1 \\ 1 & (-\lambda) \end{bmatrix} + \det \begin{bmatrix} 0 & (1 - \lambda) \\ 1 & 1 \end{bmatrix} \\ &= (1 - \lambda)(\lambda^2 - \lambda - 1) + (\lambda - 1) \\ &= (1 - \lambda)[(\lambda^2 - \lambda - 1) - 1] \\ &= (1 - \lambda)(\lambda - 2)(\lambda + 1) \end{aligned}$$

Since A has 3 distinct eigenvalues, A is diagonalizable. We need to find the eigenspaces.

To find E_1 we need to find the null space of $A - I$:

$$[A - I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Thus,

$$E_1 = \text{span} \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

and hence has basis

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

To find E_2 we need to find the null space of $A - 2I$:

$$[A - 2I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Thus,

$$E_2 = \text{span} \left(\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \right)$$

and hence has basis

$$\left\{ \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \right\}.$$

To find E_{-1} we need to find the null space of $A + I$:

$$[A + I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Thus,

$$E_{-1} = \text{span} \left(\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right) \right)$$

and so has basis

$$\left\{ \left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right) \right\}.$$

We now let P be the matrix of eigenvectors and D be the diagonal matrix whose diagonal entries are the eigenvalues (listed in the same order as the columns of P). That is:

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then A is similar to D . That is, $P^{-1}AP = D$.

(16) We begin by diagonalizing the matrix $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$.

The characteristic polynomial of A is

$$\det \begin{bmatrix} (-4 - \lambda) & 6 \\ -3 & (5 - \lambda) \end{bmatrix} = (-4 - \lambda)(5 - \lambda) + 18 = (\lambda - 2)(\lambda + 1).$$

So, A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$.

The eigenspace E_2 is the null space of $A - 2I$:

$$[A - 2I \mid \mathbf{0}] = \begin{bmatrix} -6 & 6 & \mid & 0 \\ -3 & 3 & \mid & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & \mid & 0 \\ 0 & 0 & \mid & 0 \end{bmatrix}.$$

So, E_2 has basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

The eigenspace E_{-1} is the null space of $A + I$:

$$[A + I \mid \mathbf{0}] = \begin{bmatrix} -3 & 6 & \mid & 0 \\ -3 & 6 & \mid & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & \mid & 0 \\ 0 & 0 & \mid & 0 \end{bmatrix}.$$

So, E_{-1} has basis

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

Let

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $D = P^{-1}AP \implies A = PDP^{-1}$. So,

$$\begin{aligned}
 A^9 &= PD^9P^{-1} \\
 &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^9 & 0 \\ 0 & (-1)^9 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} -514 & 1026 \\ -513 & 1025 \end{bmatrix}
 \end{aligned}$$

Section 4.6:

- (8) We know that $\lambda = 1$ is an eigenvalue of the given matrix A and that L has 3 equal columns which is the probability vector given by the eigenvector with eigenvalue 1. We need to find E_1 :

$$[A - I \mid \mathbf{0}] = \begin{bmatrix} -1/2 & 1/3 & 1/6 & | & 0 \\ 1/2 & -1/2 & 1/3 & | & 0 \\ 0 & 1/6 & -1/2 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1/2 & -1/2 & 1/3 & | & 0 \\ 0 & 1/6 & -1/2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

So, if \mathbf{x} is in the null space of $A - I$, then

$$\mathbf{x} = \begin{bmatrix} 7/3t \\ 3t \\ t \end{bmatrix}$$

for any $t \in \mathbb{R}$. Letting $t = 3$, we see that E_1 has basis

$$\left\{ \begin{bmatrix} 7 \\ 9 \\ 3 \end{bmatrix} \right\}.$$

We now turn this into a probability vector

$$\begin{bmatrix} \frac{7}{7+9+3} \\ \frac{9}{7+9+3} \\ \frac{3}{7+9+3} \end{bmatrix} = \begin{bmatrix} 7/19 \\ 9/19 \\ 3/19 \end{bmatrix}.$$

Therefore,

$$L = \begin{bmatrix} 7/19 & 7/19 & 7/19 \\ 9/19 & 9/19 & 9/19 \\ 3/19 & 3/19 & 3/19 \end{bmatrix}.$$