Math 314/814–002: Matrix Theory **Dr. S. Cooper, Fall 2008**

Name: ______________________________

Quiz 5 Solutions

This is a take-home quiz. You are allowed to use your class notes and text, but no other resources (including books, internet, or people). This is due in class on Thursday, September 25. No late submissions will be accepted.

Please write up your solutions to the following exercises. You should write legibly and fully explain your work. Staple your pages together with this page as the cover – remember to write your full name at the top.

Exercises:

• Section 3.3: # 11, 54

• Section 3.5: # 2, 6

• *True or False?* Is the following statement true or false? It it is true then carefully explain why; otherwise, provide a specific example for which the statement does not hold.

"If A is a 5×5 matrix and the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \\ 4 \end{bmatrix}$ has a unique solution, then A is invertible."

Solutions:

Section 3.3:

(11) The coefficient matrix is

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}.$$

We see that $\det(A) = 2(3) - 1(5) = 1 \neq 0$ and so A is invertible. We have

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}.$$

Therefore, by Theorem 3.7, the solution to the given system is

$$\mathbf{x} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \end{bmatrix}.$$

(54) Let A be the given matrix. We form the multi-augmented matrix $[A|I]$ and try to row-reduce this to $[I \mid A^{-1}]$. We calculate

$$[A \mid I] = \begin{bmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & | & -1/2 & 1/2 & 1/2 \end{bmatrix}.$$

We conclude that A is invertible and

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}.$$

Section 3.5:

(2) The given set

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$$

is not a subspace of $\mathbb{R}^2$. For example,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is in S since $1 \geq 0$ and $2 \geq 0$. However,

$$-2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

is not in S since $-2 < 0$ and $-4 < 0$. That is, the set S is not closed under scalar multiplication.

(6) The given set

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : z = 2x, y = 0 \right\}$$

is a subspace of $\mathbb{R}^3$. To see this we verify the three parts of the definition of a subspace:

(a) The zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in S since the second component is 0 and the third component equals twice the first.

(b) Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in S. Then $\mathbf{a}$ and $\mathbf{b}$ are of the form

$$\mathbf{a} = \begin{bmatrix} s \\ 0 \\ 2s \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix}$$

for some $s, t \in \mathbb{R}$. Thus

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} s+t \\ 0 \\ 2s+2t \end{bmatrix}.$$

We see that the second component of $\mathbf{a} + \mathbf{b}$ equals 0 and the third component is twice the first component. Thus, $\mathbf{a} + \mathbf{b}$ is in S. That is, the set S is closed under vector addition.

(c) Let $d \in \mathbb{R}$ and $\mathbf{a}$ be a vector in S. Then $\mathbf{a}$ is of the form

$$\mathbf{a} = \begin{bmatrix} s \\ 0 \\ 2s \end{bmatrix}$$

for some $s \in \mathbb{R}$. Thus,

$$d\mathbf{x} = \begin{bmatrix} ds \\ 0 \\ 2ds \end{bmatrix}.$$

We see that the second component of $d\mathbf{a}$ is zero and the third component of $d\mathbf{a}$ is twice the first component of $d\mathbf{a}$. So, $d\mathbf{a}$ is in the set S. Therefore, S is closed under scalar multiplication.

True or False?

The statement is true. First note that the matrix A is square. If $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \\ 4 \end{bmatrix}$ has a unique solution, then every column of $RREF(A)$ must have a leading one (no free variables existed in the given system). This means that $RREF(A) = I$. So, by the Fundamental Theorem of Invertible Matrices, A^{-1} exists.