Math 314/814–006: Matrix Theory **Dr. S. Cooper, Fall 2008**

Name: ____

Quiz 4 Solutions

Please write your solutions to the following exercises in the space provided. You should write legibly and fully explain your work.

Good Luck!

(1) Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & 5 \\ 10 & 24 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 6 & 10 \end{bmatrix}$.
Write B as a linear combination of A_1, A_2 and A_3 by solving a system of linear equations. [10 pts]

Solution: We need to find scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$B = \begin{bmatrix} -1 & 3 \\ 6 & 10 \end{bmatrix} = c_1A_1 + c_2A_2 + c_3A_3 = \begin{bmatrix} c_1 + 2c_3 & c_2 + 5c_3 \\ 2c_2 + 10c_3 & 2c_1 + 4c_2 + 24c_3 \end{bmatrix}.$$

Thus, we need to solve the system

$$\begin{aligned} c_1 + 2c_3 &= -1 \\ c_2 + 5c_3 &= 3 \\ 2c_2 + 10c_3 &= 6 \\ 2c_1 + 4c_2 + 24c_3 &= 10 \end{aligned}$$

We row reduce the augmented matris of this system

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 5 & 3 \\ 0 & 2 & 10 & 6 \\ 2 & 4 & 24 & 10 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 5 & 3 \\ 0 & 2 & 10 & 6 \\ 0 & 4 & 20 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We let $c_3 = t \in \mathbb{R}$. Then

$$\begin{aligned} c_2 &= -5c_3 + 3 = -5t + 3 \\ c_1 &= -2c_3 - 1 = -2t - 1 \end{aligned}$$

Letting $t = 1$, we have

$$B = -3A_1 - 2A_2 + A_3.$$

(2) Let $A = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$. Compute AB. [5 pts]

Solution:

$$\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -6 & 3 \\ -4 & 12 & 14 \end{bmatrix}.$$

(3) Suppose that $\mathbf{u}, \mathbf{v}$ are linearly independent vectors. Show that the vectors $\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}$ are also linearly independent. [5 pts]

Solution: Suppose that for some scalars $c_1, c_2 \in \mathbb{R}$ we have the equation

$$c_1(\mathbf{u}+\mathbf{v}) + c_2(\mathbf{u}-\mathbf{v}) = \mathbf{0}.$$

Then, rearranging this equation, we see that

$$(c_1+c_2)\mathbf{u} + (c_1-c_2)\mathbf{v} = \mathbf{0}.$$

Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, it must be the case that

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 0 \end{aligned}$$

This system implies that $c_1 = -c_2$ and $c_1 = c_2$. However, this can only happen if $c_1 = c_2 = 0$. So, by definition, $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ must also be linearly independent.