Math 314/814–006: Matrix Theory **Dr. S. Cooper, Fall 2008**

Name: ____

Quiz 11 Solutions

Please write your solutions to the following exercises in the space provided. You should write legibly and fully explain your work.

Good Luck!

(1) Let $W = span\left(\begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}\right)$. Find a basis for $W^{\perp}$. [6 pts]

Solution: We have that

$$W = span\left(\begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}\right) = col(A)$$

where

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 3 & -2 \\ -2 & 1 \end{bmatrix}.$$

So, by Theorem 5.10, $W^{\perp} = null(A^T)$. We have

$$A^T = \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

Thus, when we solve the system $A^T\mathbf{x} = \mathbf{0}$, we have $x_1 = x_2 - 3x_3 + 2x_4 = -s + t, x_2 = 2x_3 - x_4 = 2s - t, x_3 = s, x_4 = t$. So, a basis for $W^{\perp} = null(A^T)$ is

$$\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

(2) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u_1} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, and $\mathbf{u_2} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$. Find the orthogonal projection of $\mathbf{v}$ onto the subspace $W = span(\mathbf{u_1}, \mathbf{u_2})$. [7 pts]

Solution: By definition,

$$proj_W(\mathbf{v}) = \left(\frac{\mathbf{u_1} \cdot \mathbf{v}}{\mathbf{u_1} \cdot \mathbf{u_1}}\right)\mathbf{u_1} + \left(\frac{\mathbf{u_2} \cdot \mathbf{v}}{\mathbf{u_2} \cdot \mathbf{u_2}}\right)\mathbf{u_2}.$$

We calculate

$$\begin{aligned} \mathbf{u_1} \cdot \mathbf{v} &= 1 \\ \mathbf{u_2} \cdot \mathbf{v} &= 13 \\ \mathbf{u_1} \cdot \mathbf{u_1} &= 9 \\ \mathbf{u_2} \cdot \mathbf{u_2} &= 18 \end{aligned}$$

Thus,

$$proj_W(\mathbf{v}) = \frac{1}{9}\mathbf{u_1} + \frac{13}{18}\mathbf{u_2} = \begin{bmatrix} -1/2 \\ 1/2 \\ 3 \end{bmatrix}.$$

(3) A susbpace W of $\mathbb{R}^3$ has basis vectors $\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$. Apply the Gram-Schmidt Process to find an orthogonal basis for W. [7 pts]

Solution: Let $\mathbf{v_1} = \mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Then let

$$\begin{aligned} \mathbf{v_2} &= \mathbf{x_2} - \left(\frac{\mathbf{v_1} \cdot \mathbf{x_2}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1} \\ &= \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1/2 \\ 1/2 \\ 2 \end{bmatrix} \end{aligned}$$

Then the set $\{\mathbf{v_1}, \mathbf{v_2}\}$ is an orthogonal basis for W.