Problem Set 11
Due: Thursday, April 21

Work all of the following problems. Remember, you are encouraged to work together on Problem Sets, but each student must turn in his or her own write-up. Be sure to adhere to the Rules and Expectations outlined in the Course Information Sheet.

1 Traditional Problems

1. (Gallian, Chapter 10 Exercises, #8) Let G be a group of permutations. For each σ in G, define

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

   Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel? Why does this homomorphism allow you to conclude that A_n is a normal subgroup of S_n of index 2?

2. (Gallian, Chapter 10 Exercises, #12) Suppose that k is a divisor of n. Prove that $\mathbb{Z}_n/\langle k\rangle \approx \mathbb{Z}_k$.

3. (Gallian, Chapter 10 Exercises, #15) Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and Ker $\phi = \{0, 10, 20\}$. If $\phi(23) = 9$, determine all elements that map to 9.

4. (Gallian, Chapter 10 Exercises, #29) Suppose that there is a homomorphism from a finite group G onto $\mathbb{Z}_{10}$. Prove that G has normal subgroups of indexes 2 and 5.

5. (Gallian, Chapter 10 Exercises, #40) (Third Isomorphism Theorem) If M and N are normal subgroups of G and $N \leq M$, prove that $(G/N)/(M/N) \approx G/M$.

6. (Gallian, Chapter 11 Exercises, #9) Suppose that G is an Abelian group of order 120 and that G has exactly three elements of order 2. Determine the isomorphism class of G.

7. (Gallian, Chapter 11 Exercises, #25) Let

$$G = \{1, 7, 43, 49, 51, 57, 93, 99, 101, 107, 143, 149, 151, 157, 193, 199\}$$

   under multiplication modulo 200. Express G as an external and an internal direct product of cyclic groups.

8. (Gallian, Chapter 11 Exercises, #28) Suppose that G is an Abelian group of order 16, and in computing the orders of its elements, you come across an element of order 8 and two elements of order 2. Explain why no further computations are needed to determine the isomorphism class of G.