

Problem Set 10

Due: Thursday, April 14

Work all of the following problems. Remember, you are encouraged to work together on Problem Sets, but each student must turn in his or her own write-up. Be sure to adhere to the Rules and Expectations outlined in the Course Information Sheet.

1 Traditional Problems

- (Gallian, Chapter 9 Exercises, #10) Prove that a factor group of a cyclic group is cyclic.
- (Gallian, Chapter 9 Exercises, #11) Let H be a normal subgroup of G . If H and G/H are Abelian, must G be Abelian? Either prove the statement or give a concrete example to show it is not true in general.
- (Gallian, Chapter 9 Exercises, #12) Prove that a factor group of an Abelian group is Abelian.
- (Gallian, Chapter 9 Exercises, #34) In \mathbb{Z} , let $H = \langle 5 \rangle$ and $K = \langle 7 \rangle$. Prove that $\mathbb{Z} = HK$. Does $\mathbb{Z} = H \times K$?
- (Gallian, Chapter 9 Exercises, #48) If $|G| = pq$, where p and q are primes that are not necessarily distinct, prove that $|Z(G)| = 1$ or pq .
- (Gallian, Chapter 9 Exercises, #52) Let $G = \{-1, 1, -i, i, -j, j, -k, k\}$, where $i^2 = j^2 = k^2 = -1$, $-i = (-1)i$, $1^2 = (-1)^2 = 1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. See page 196 of the text for a visual description of these multiplication rules. The group G is called the group of *quaternions* and is used in many applications of group theory.
 - Construct the Cayley table for G .
 - Show that $H = \{1, -1\}$ is a normal subgroup of G .
 - Construct the Cayley table for G/H . Is G/H isomorphic to \mathbb{Z}_4 or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$?
- (Gallian, Chapter 9 Exercises, #55) Let N be a normal subgroup of G and let H be any subgroup of G . Prove that NH is a subgroup of G . Give an example to show that NH need not be a subgroup of G if neither N nor H is normal.
- (Gallian, Chapter 9 Exercises, #60) Let G be a group and let G' be the subgroup of G generated by the set $S = \{x^{-1}y^{-1}xy \mid x, y \in G\}$. The subgroup G' is called the *derived subgroup* or the *commutator subgroup* of G .
 - Prove that G' is normal in G .
 - Prove that G/G' is Abelian.
 - If G/N is Abelian, prove that $G' \leq N$.
 - Prove that if H is a subgroup of G and $G' \leq H$, then H is normal in G .

2 Computer Problems

As outlined on Problem Set 0, please intersperse your GAP commands and output with your explanations. You should create a log file as described in Chapter –1 of the lab manual. If you type up your solutions, you can cut and paste from this log file into your solution file; please use a different font so it is easy to tell what is what. If you hand-write your solutions, you should still print out your log file; then physically cut and paste it into your solutions.

Complete the following problems from Chapter 9 of the computer lab manual. You do not have to find the Cayley tables by hand - use GAP.

1. Problem 9.1.
2. Problem 9.2.
3. Problem 9.3.
4. Problem 9.4.
5. Problem 9.5.

Note that the file “orderFrequency” is described in the Appendix to Chapter 4 (see page 25 of the manual).