Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of October 7

Note: The exercises from Section 4.3 should be completed the week of October 14.

Section 4.2:

(47) $\det(AB) = \det(A)\det(B) = (3)(-2) = -6$

(48) $\det(A^2) = \det(A)\det(A) = (3)(3) = 9$

(49) $\det(B^{-1}A) = \det(B^{-1})\det(A) = \dfrac{1}{\det(B)}\det(A) = -\dfrac{1}{2}(3) = -\dfrac{3}{2}$

(51) $\det(3B^T) = 3^n\det(B^T) = 3^n\det(B) = 3^n(-2) = (-2)3^n$

(53) $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$

(55) If A is idempotent then, by definition, $A^2 = A$. Thus

$$\det(A) = \det(A^2) = \det(A)\det(A).$$

Thus,

$$\det(A) - \det(A)\det(A) = 0 \implies \det(A)[1 - \det(A)] = 0.$$

We conclude that $\det(A) = 0$ or $\det(A) = 1$.

(59) The given system is equivalent to $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We have

$$A_1(\mathbf{b}) = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, A_3(\mathbf{b}) = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We find

$$\det(A_1(\mathbf{b})) = -2,\ \ \det(A_2(\mathbf{b})) = 0,\ \ \det(A_3(\mathbf{b})) = 2,\ \ \det(A) = 2.$$

So, by Cramer's Rule,

$$\begin{aligned} x &= \frac{\det(A_1(\mathbf{b}))}{\det(A)} = -1 \\ y &= \frac{\det(A_2(\mathbf{b}))}{\det(A)} = 0 \\ z &= \frac{\det(A_3(\mathbf{b}))}{\det(A)} = 1 \end{aligned}$$

This gives the unique solution,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

(63) We first calculate $\det(A) = 2$.

Using the notation from class and the text book, we let C_{ij} be the (i,j)-cofactor of A. We find

$$C_{11} = 1,\ \ C_{12} = 0,\ \ C_{13} = 0$$

$$C_{21} = -1,\ \ C_{22} = 2,\ \ C_{23} = 0$$

$$C_{31} = -2,\ \ C_{32} = -2,\ \ C_{33} = 2$$

Let

$$C = [C_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & -2 & 2 \end{bmatrix}.$$

Then

$$adj\ A = C^T = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore,

$$A^{-1} = \frac{1}{\det(A)} adj\ A = \frac{1}{2} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Section 4.3:

(1) (a) The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{bmatrix}(1-\lambda) & 3\\ -2 & (6-\lambda)\end{bmatrix}\\ &= (1-\lambda)(6-\lambda)+6\\ &= \lambda^2-7\lambda+12\\ &= (\lambda-4)(\lambda-3)\end{aligned}$$

(b) The eigenvalues of A are the roots of $\det(A-\lambda I)=0$: $\lambda_1=4$ and $\lambda_2=3$.

(c) To find a basis for E_4, we find the null space of $(A-4I)$:

$$[A-4I \mid \mathbf{0}] = \left[\begin{array}{cc|c}-3 & 3 & 0\\ -2 & 2 & 0\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right].$$

Thus,

$$E_4 = span\left(\begin{bmatrix}1\\1\end{bmatrix}\right).$$

A basis for E_4 is

$$\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

We repeat the above process to find a basis for E_3, i.e. we find the null space of $(A-3I)$:

$$[A-3I \mid \mathbf{0}] = \left[\begin{array}{cc|c}-2 & 3 & 0\\ -2 & 3 & 0\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}2 & -3 & 0\\ 0 & 0 & 0\end{array}\right].$$

Thus,

$$E_3 = span\left(\begin{bmatrix}3/2\\1\end{bmatrix}\right).$$

A basis for E_3 is

$$\left\{\begin{bmatrix}3\\2\end{bmatrix}\right\}.$$

(d) The algebraic multiplicity for $\lambda_1 = 4$ is 1, and the geometric multiplicity for $\lambda_1 = 4$ is 1. Similarly, the algebraic multiplicity for $\lambda_2 = 3$ is 1, and the geometric multiplicity for $\lambda_2 = 3$ is 1.

(2) (a) The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} (2-\lambda) & 1 \\ -1 & (-\lambda) \end{bmatrix} \\ &= (-\lambda)(2-\lambda) + 1 \\ &= \lambda^2 - 2\lambda + 1 \\ &= (\lambda - 1)^2 \end{aligned}$$

(b) The eigenvalues of A are the roots of $\det(A - \lambda I) = 0$. So, the only eigenvalue of A is $\lambda_1 = 1$.

(c) To find a basis for E_1, we find the null space of $(A - 1I)$:

$$[A - 1I \mid \mathbf{0}] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus,

$$E_1 = span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right).$$

A basis for E_1 is

$$\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$$

(d) The algebraic multiplicity for $\lambda_1 = 1$ is 2, and the geometric multiplicity for $\lambda_1 = 1$ is 1.

(5) (a) The characteristic polynomial of A is

$$\begin{aligned}
\det(A-\lambda I) &= \det\begin{bmatrix} (1-\lambda) & 2 & 0 \\ -1 & (-1-\lambda) & 1 \\ 0 & 1 & (1-\lambda) \end{bmatrix} \\
&= (1-\lambda)\det\begin{bmatrix} (-1-\lambda) & 1 \\ 1 & (1-\lambda) \end{bmatrix} - 2\det\begin{bmatrix} -1 & 1 \\ 0 & (1-\lambda) \end{bmatrix} \\
&= (1-\lambda)[(-1-\lambda)(1-\lambda)-1] - 2(-1)(1-\lambda) \\
&= -\lambda^3+\lambda^2 \\
&= -\lambda^2(\lambda-1)
\end{aligned}$$

(b) The eigenvalues of A are the roots of $\det(A-\lambda I)=0$: $\lambda_1=0$ and $\lambda_2=1$.

(c) To find a basis for E_0, we find the null space of $(A-0I)$:

$$[A-0I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus,

$$E_0 = span\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right).$$

A basis for E_0 is

$$\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}.$$

We repeat the above process to find a basis for E_1, i.e. we find the null space of $(A-1I)$:

$$[A-1I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus,

$$E_1 = span\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

A basis for E_1 is

$$\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

(d) The algebraic multiplicity for $\lambda_1 = 0$ is 2, and the geometric multiplicity for $\lambda_1 = 0$ is 1. The algebraic multiplicity for $\lambda_2 = 1$ is 1, and the geometric multiplicity for $\lambda_2 = 1$ is 1.

(8) (a) The characteristic polynomial of A is

$$\begin{aligned}
\det(A - \lambda I) &= \det\begin{bmatrix} (1-\lambda) & -1 & -1 \\ 0 & (2-\lambda) & 0 \\ -1 & -1 & (1-\lambda) \end{bmatrix} \\
&= (2-\lambda)\det\begin{bmatrix} (1-\lambda) & -1 \\ -1 & (1-\lambda) \end{bmatrix} \\
&= (2-\lambda)[(1-\lambda)(1-\lambda) - 1] \\
&= (2-\lambda)(\lambda^2 - 2\lambda) \\
&= -\lambda(\lambda-2)^2
\end{aligned}$$

(b) The eigenvalues of A are the roots of $\det(A - \lambda I) = 0$: $\lambda_1 = 0$ and $\lambda_2 = 2$.

(c) To find a basis for E_0, we find the null space of $(A - 0I)$:

$$[A - 0I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus,

$$E_0 = span\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

A basis for E_0 is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We repeat the above process to find a basis for E_2, i.e. we find the null space of $(A - 2I)$:

$$[A - 2I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus,

$$E_2 = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

A basis for E_2 is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(d) The algebraic multiplicity for $\lambda_1 = 0$ is 1, and the geometric multiplicity for $\lambda_1 = 0$ is 1. The algebraic multiplicity for $\lambda_2 = 2$ is 2, and the geometric multiplicity for $\lambda_2 = 2$ is 2.