Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of September 30

Note: The exercises from Section 4.2 should be completed the week of October 7.

Section 4.1:

(5) We have

$$A\mathbf{v} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3\mathbf{v}.$$

So, by definition, $\mathbf{v}$ is an eigenvector of A with eigenvalue $\lambda = 3$.

(6) We have

$$A\mathbf{v} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathbf{v}.$$

So, by definition, $\mathbf{v}$ is an eigenvector of A with eigenvalue $\lambda = 0$.

(9) We want to find $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = 1\mathbf{x}$. Equivalently, we want to find $\mathbf{x} \neq \mathbf{0}$ such that

$$A\mathbf{x} - 1\mathbf{x} = (A - 1I)\mathbf{x} = \mathbf{0}.$$

So, we need to find the null space of $A - 1I$. We form the associated augmented matrix and row reduce:

$$[A - I \mid \mathbf{0}] = \left[\begin{array}{cc|c} -1 & 4 & 0 \\ -1 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Solving the system, we have

$$null(A - 1I) = \left\{ t \begin{bmatrix} 4 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Any *non-zero* multiple of $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 1$. In particular,

$$A\mathbf{u} = 1\mathbf{u}$$

which shows that $\mathbf{u}$ is an eigenvector of A with eigenvalue 1.

(10) We want to find $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = 4\mathbf{x}$. Equivalently, we want to find $\mathbf{x} \neq \mathbf{0}$ such that

$$A\mathbf{x} - 4\mathbf{x} = (A - 4I)\mathbf{x} = \mathbf{0}.$$

So, we need to find the null space of $A - 4I$. We form the associated augmented matrix and row reduce:

$$[A - 4I \mid \mathbf{0}] = \begin{bmatrix} -4 & 4 & | & 0 \\ -1 & 1 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

Solving the system, we have

$$null(A - 4I) = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Any *non-zero* multiple of $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 4$. In particular,

$$A\mathbf{u} = 4\mathbf{u}$$

which shows that $\mathbf{u}$ is an eigenvector of A with eigenvalue 4.

(11) We want to find $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = -1\mathbf{x}$. Equivalently, we want to find $\mathbf{x} \neq \mathbf{0}$ such that

$$A\mathbf{x} - (-1)\mathbf{x} = (A + I)\mathbf{x} = \mathbf{0}.$$

So, we need to find the null space of $A + I$. We form the associated augmented matrix and row reduce:

$$[A + I \mid \mathbf{0}] = \begin{bmatrix} 2 & 0 & 2 & | & 0 \\ -1 & 2 & 1 & | & 0 \\ 2 & 0 & 2 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Solving the system, we have

$$null(A + I) = \left\{ t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Any *non-zero* multiple of $\mathbf{u} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = -1$. In particular,

$$A\mathbf{u} = -1\mathbf{u}$$

which shows that $\mathbf{u}$ is an eigenvector of A with eigenvalue -1.

(13) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is the matrix of a reflection F in the y-axis. The only vectors that F maps parallel to themselves are vectors of the form $\begin{bmatrix} 0 \\ s \end{bmatrix}$ and $\begin{bmatrix} t \\ 0 \end{bmatrix}$. Vectors of the form $\mathbf{u} = \begin{bmatrix} 0 \\ s \end{bmatrix}$ are transformed to $F(\mathbf{u}) = \begin{bmatrix} 0 \\ s \end{bmatrix} = 1\mathbf{u}$ (i.e. these are eigenvectors with eigenvalue 1). Vectors of the form $\mathbf{v} = \begin{bmatrix} t \\ 0 \end{bmatrix}$ are transformed to $F(\mathbf{v}) = \begin{bmatrix} -t \\ 0 \end{bmatrix} = -1\mathbf{v}$ (i.e. these are eigenvectors with eigenvalue -1). Therefore, we have the eigenspaces

$$E_1 = span\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

and

$$E_{-1} = span\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

(17) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is the matrix of the transformation T which stretches by a factor of 2 horizontally and a factor of 3 vertically. The only vectors that T maps parallel to themselves are vectors of the form $\begin{bmatrix} s \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ t \end{bmatrix}$. Vectors of the form $\mathbf{u} = \begin{bmatrix} s \\ 0 \end{bmatrix}$ are transformed to $T(\mathbf{u}) = \begin{bmatrix} 2s \\ 0 \end{bmatrix} = 2\mathbf{u}$ (i.e. these are eigenvectors with eigenvalue 2). Vectors of the form $\mathbf{v} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ are transformed to

$T(\mathbf{v}) = \begin{bmatrix} 0 \\ 3t \end{bmatrix} = 3\mathbf{v}$ (i.e. these are eigenvectors with eigenvalue 3). Therefore, we have the eigenspaces

$$E_2 = span\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

and

$$E_3 = span\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

(25) We first find the eigenvalues for A:

$$\det(A - \lambda I) = \det \begin{bmatrix} (2-\lambda) & 5 \\ 0 & (2-\lambda) \end{bmatrix} = (2-\lambda)^2.$$

The eigenvalues are the roots of $(2-\lambda)^2 = 0$. So, the only eigenvalue of A is $\lambda = 2$.

To find the eigenspace E_2, we need to find the null space of $A - 2I$. We row reduce the associated augmented matrix:

$$[A - 2I \mid \mathbf{0}] = \begin{bmatrix} 0 & 5 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

The solution set is E_2. We see that

$$E_2 = span\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right),$$

and so $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is a basis for the eigenspace E_2.

(26) We first find the eigenvalues for A:

$$\det(A - \lambda I) = \det \begin{bmatrix} (1-\lambda) & 2 \\ -2 & (3-\lambda) \end{bmatrix} = (1-\lambda)(3-\lambda) + 4 = \lambda^2 - 4\lambda + 7.$$

The eigenvalues are the roots of $\lambda^2 - 4\lambda + 7 = 0$. Since there are no real roots of this polynomial, A has no eigenvalues.

(27) We first find the eigenvalues for A:

$$\det(A-\lambda I)=\det\begin{bmatrix}(1-\lambda) & 1\\ -1 & (1-\lambda)\end{bmatrix}=(1-\lambda)(1-\lambda)+1=\lambda^2-2\lambda+2.$$

The eigenvalues are the roots of $\lambda^2-2\lambda+2=0$. We see that A has eigenvalues $\lambda_1=1+i$ and $\lambda_2=1-i$.

To find the eigenspace E_{1+i}, we need to find the null space of $A-(1+i)I$. We row reduce the associated augmented matrix:

$$[A-(1+i)I \mid \mathbf{0}]=\left[\begin{array}{cc|c}-i & 1 & 0\\ -1 & -i & 0\end{array}\right]\to\left[\begin{array}{cc|c}1 & i & 0\\ 0 & 0 & 0\end{array}\right].$$

The solution set is E_{1+i}. We see that

$$E_{1+i}=\mathit{span}\left(\begin{bmatrix}-i\\ 1\end{bmatrix}\right)=\mathit{span}\left(\begin{bmatrix}1\\ i\end{bmatrix}\right),$$

and so $\left\{\begin{bmatrix}1\\ i\end{bmatrix}\right\}$ is a basis for the eigenspace E_{1+i}.

To find the eigenspace E_{1-i}, we need to find the null space of $A-(1-i)I$. We row reduce the associated augmented matrix:

$$[A-(1-i)I \mid \mathbf{0}]=\left[\begin{array}{cc|c}i & 1 & 0\\ -1 & i & 0\end{array}\right]\to\left[\begin{array}{cc|c}1 & -i & 0\\ 0 & 0 & 0\end{array}\right].$$

The solution set is E_{1-i}. We see that

$$E_{1-i}=\mathit{span}\left(\begin{bmatrix}i\\ 1\end{bmatrix}\right)=\mathit{span}\left(\begin{bmatrix}1\\ -i\end{bmatrix}\right),$$

and so $\left\{\begin{bmatrix}1\\ -i\end{bmatrix}\right\}$ is a basis for the eigenspace E_{1-i}.

(28) We first find the eigenvalues for A:

$$\det(A-\lambda I)=\det\begin{bmatrix}(2-\lambda) & -3\\ 1 & (-\lambda)\end{bmatrix}=(2-\lambda)(-\lambda)+3=\lambda^2-2\lambda+3.$$

The eigenvalues are the roots of $\lambda^2 - 2\lambda + 3 = 0$. We see that A has eigenvalues $\lambda_1 = 1 + \sqrt{2}i$ and $\lambda_2 = 1 - \sqrt{2}i$.

To find the eigenspace $E_{1+\sqrt{2}i}$, we need to find the null space of $A - (1 + \sqrt{2}i)I$. We row reduce the associated augmented matrix:

$$[A-(1+\sqrt{2}i)I \mid \mathbf{0}] = \begin{bmatrix} (1-\sqrt{2}i) & -3 & | & 0 \\ 1 & (-1-\sqrt{2}i) & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & (-1-\sqrt{2}i) & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

The solution set is $E_{1+\sqrt{2}i}$. We see that

$$E_{1+\sqrt{2}i} = span\left(\begin{bmatrix} 1+\sqrt{2}i \\ 1 \end{bmatrix}\right),$$

and so $\left\{\begin{bmatrix} 1+\sqrt{2}i \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace $E_{1+\sqrt{2}i}$.

To find the eigenspace $E_{1-\sqrt{2}i}$, we need to find the null space of $A - (1 - \sqrt{2}i)I$. We row reduce the associated augmented matrix:

$$[A-(1-\sqrt{2}i)I \mid \mathbf{0}] = \begin{bmatrix} (1+\sqrt{2}i) & -3 & | & 0 \\ 1 & (-1+\sqrt{2}i) & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & (-1+\sqrt{2}i) & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

The solution set is $E_{1-\sqrt{2}i}$. We see that

$$E_{1-\sqrt{2}i} = span\left(\begin{bmatrix} 1-\sqrt{2}i \\ 1 \end{bmatrix}\right),$$

and so $\left\{\begin{bmatrix} 1-\sqrt{2}i \\ -i \end{bmatrix}\right\}$ is a basis for the eigenspace $E_{1-\sqrt{2}i}$.

Section 4.2:

(1) Using cofactor expansion along the first row, we have

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} &= 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 5 & 1 \\ 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} \\ &= 1(2-1) - 0 + 3(5-0) \\ &= 16 \end{aligned}$$

Using cofactor expansion along the first column, we have

$$\begin{aligned}\begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} &= 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 5\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} \\ &= 1(2-1) - 5(0-3) + 0 \\ &= 16\end{aligned}$$

(6) Using cofactor expansion along the first row, we have

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1(45-48) - 2(36-42) + 3(32-35) \\ &= 0\end{aligned}$$

Using cofactor expansion along the first column, we have

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \\ &= 1(45-48) - 4(18-24) + 7(12-15) \\ &= 0\end{aligned}$$

(9) Using cofactor expansion along the third row, we have

$$\begin{aligned}\begin{vmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{vmatrix} &= 1\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} - (-1)\begin{vmatrix} -4 & 3 \\ 2 & 4 \end{vmatrix} + 0\begin{vmatrix} -4 & 1 \\ 2 & -2 \end{vmatrix} \\ &= 1(4+6) + 1(-16-6) + 0 \\ &= -12\end{aligned}$$

(13) Starting with cofactor expansion along the third row, we have

$$\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{vmatrix} = -1 \begin{vmatrix} 1 & 0 & 3 \\ 2 & 2 & 6 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= -1\left(1\begin{vmatrix} 2 & 6 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}\right)$$

$$= -1[(2-12) - 0 + 3(4-2)]$$

$$= -1(-4)$$

$$= 4$$

(23) We row reduce the given matrix A keeping track of our row operations along the way:

$$A = \begin{bmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 4 \\ -4 & 1 & 3 \end{bmatrix}$$

$$\rightarrow C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 4 \\ 0 & -3 & 3 \end{bmatrix}$$

$$\rightarrow D = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

By Theorem 4.2, we have:

$$\det(B) = -\det(A); \det(C) = \det(B); \det(D) = -\det(C).$$

So,

$$\det(A) = -\det(B) = -\det(C) = \det(D) = 1\begin{vmatrix} -3 & 3 \\ 0 & 4 \end{vmatrix} = -12.$$

(26) We can row reduce the given matrix A with the operation $R_3 \rightarrow R_3 - 2R_1$. This gives a new matrix B such that $\det(B) = \det(A)$. Moreover, since the third row of B is a zero row, $\det(B) = 0$. We conclude that $\det(A) = 0$.

(35) By Theorem 4.3,

$$\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(4) = 8.$$

(36) By Theorem 4.10,

$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = 4.$$

So, by Theorem 4.3,

$$\begin{vmatrix} 3a & 3d & 3g \\ -b & -e & -h \\ 2c & 2f & 2i \end{vmatrix} = 4(3)(-1)(2) = -24.$$

We again apply Theorem 4.10 to obtain

$$\begin{vmatrix} 3a & -b & 2c \\ 3d & -e & 2f \\ 3g & -h & 2i \end{vmatrix} = -24.$$

(37) By Theorem 4.3,

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -4.$$

(40) Observe that we have the following row reductions:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \to B = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix} \to C = \begin{bmatrix} a & b & c \\ 2d-3g & 2e-3h & 2f-3i \\ g & h & i \end{bmatrix}.$$

By Theorem 4.3,

$$\det(C) = \det(B) = 2\det(A) = 2(4) = 8.$$

(45) Using cofactor expansion along the first column of A, we compute

$$\begin{aligned}
\det(A) &= k\begin{vmatrix} (k+1) & 1 \\ -8 & (k-1) \end{vmatrix} - 0\begin{vmatrix} -k & 3 \\ -8 & (k-1) \end{vmatrix} + k\begin{vmatrix} -k & 3 \\ (k+1) & 1 \end{vmatrix} \\
&= k(k^2+7) + k(-4k-3) \\
&= k^3 - 4k^2 + 4k \\
&= k(k^2 - 4k + 4) \\
&= k(k-2)^2
\end{aligned}$$

By Theorem 4.6, A is invertible if and only if $\det(A) \neq 0$. We conclude that A is invertible if and only if $k \neq 0, 2$.