Math 314/814: Matrix Theory **Dr. S. Cooper, Fall 2008**

Homework Solutions – Week of November 11

Section 6.2:

(3) Let c_1, c_2, c_3, c_4 be scalars such that

$$c_1 \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix} + c_4 \begin{bmatrix} -1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Comparing entries of the matrices, this leads to the linear system of equations:

$$\begin{aligned} -c_1 + 3c_2 - c_4 &= 0 \\ c_1 + 2c_3 &= 0 \\ -2c_1 + c_2 - 3c_3 - c_4 &= 0 \\ 2c_1 + c_2 + c_3 + 7c_4 &= 0 \end{aligned}$$

We form the associated augmented matrix and row-reduce:

$$\left[\begin{array}{rrrr|r} -1 & 3 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ -2 & 1 & -3 & -1 & 0 \\ 2 & 1 & 1 & 7 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We see that there is a free variable, and so we conclude that the given matrices are linearly dependent. To find a dependence relation we need to solve the system. Let $c_4 = t$. Then $c_3 = 2t, c_2 = -t$ and $c_1 = -4t$. Letting $t = 1$ and subbing in for the scalars in the initial matrix equation we see that

$$\begin{bmatrix} -1 & 0 \\ -1 & 7 \end{bmatrix} = 4 \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} - 2 \begin{bmatrix} -1 & 0 \\ -1 & 7 \end{bmatrix}.$$

(5) Suppose we have scalars c_1 and c_2 such that

$$c_1(x) + c_2(1 + x) = 0 + 0x.$$

By rearranging this is equivalent to

$$c_2 + (c_1 + c_2)x = 0 + 0x.$$

Comparing coefficients we see that $c_2 = 0$ and hence $c_1 = 0$. Thus, the given polynomials are linearly independent.

(6) Suppose we have scalars c_1, c_2 and c_3 such that

$$c_1(1+x) + c_2(1+x^2) + c_3(1-x+x^2) = 0 + 0x + 0x^2.$$

By rearranging this is equivalent to

$$(c_1 + c_2 + c_3) + (c_1 - c_3)x + (c_2 + c_3)x^2 = 0 + 0x + 0x^2.$$

Comparing coefficients we obtain the linear system of equations

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 - c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

We form the associated augmented matrix and row-reduce:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Since there are no free variables, we have only the trivial solution $c_1 = c_2 = c_3 = 0$. Therefore, the given polynomials are linearly independent.

(10) The only scalars which solve the linear combination

$$c_1(1) + c_2(\sin x) + c_2(\cos x) = 0$$

are $c_1 = c_2 = c_3 = 0$. Thus, the given functions are linearly independent.

(11) Using the trig identity $\sin^2 x + \cos^2 x = 1$, we have the linear combination

$$1 - \sin^2 x - \cos^2 x = 0.$$

Therefore, the given functions are linearly dependent. One possible linear dependence relation is

$$1 = \sin^2 x + \cos^2 x.$$

(17) (a) The set $\{\mathbf{u}, +\mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{u} + \mathbf{w}\}$ is linearly independent. To see this, suppose we have scalars c_1, c_2 and c_3 such that

$$c_1(\mathbf{u}, +\mathbf{v}) + c_2(\mathbf{v} + \mathbf{w}) + c_3(\mathbf{u} + \mathbf{w}) = \mathbf{0}.$$

Rearranging this equation we obtain

$$(c_1 + c_3)\mathbf{u} + (c_1 + c_2)\mathbf{v} + (c_2 + c_3)\mathbf{w} = \mathbf{0}.$$

Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set we must have the linear system:

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

We form the associated augmented matrix and row-reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Since there are no free variables, we have only the trivial solution $c_1 = c_2 = c_3 = 0$.

(b) No, the set $\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{u} - \mathbf{w}\}$ does not have to be linearly independent. For example, let $V = \mathbb{R}^3$ and let $\mathbf{u} = \mathbf{e_1}, \mathbf{v} = \mathbf{e_2}$ and $\mathbf{w} = \mathbf{e_3}$. Then

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v} - \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{u} - \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Observe the dependence relation

$$\mathbf{u} - \mathbf{w} = (\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w}).$$

(19) Let c_1, c_2, c_3, c_4 be scalars such that

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Comparing entries of the matrices, this leads to the linear system of equations:

$$\begin{aligned} c_1 + c_3 + c_4 &= 0 \\ -c_2 + c_3 + c_4 &= 0 \\ c_2 + c_3 + c_4 &= 0 \\ c_1 + c_3 - c_4 &= 0 \end{aligned}$$

We form the associated augmented matrix and row-reduce:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Thus, $c_1 = c_2 = c_3 = c_4 = 0$. That is, the matrices in $\mathcal{B}$ are linearly independent. Since $\dim(M_{22}) = 4$ and $\mathcal{B}$ consists of 4 linearly independent matrices in M_{22}, Theorem 6.10 (c) says that $\mathcal{B}$ is a basis for M_{22}.

(22) Suppose that c_1, c_2, c_3 are scalars such that

$$c_1 x + c_2(1+x) + c_3(x - x^2) = 0 + 0x + 0x^2.$$

Rearranging we see that this equation is equivalent to

$$c_2 + (c_1 + c_2 + c_3)x - c_3 x^2 = 0 + 0x + 0x^2.$$

Comparing coefficients we see immediately that $c_2 = c_3 = 0$. This implies that $c_1 = 0$. Thus, the polynomials in $\mathcal{B}$ are linearly independent. Since $\dim(\mathcal{P}_2) = 3$ and $\mathcal{B}$ consists of 3 linearly independent polynomials in $\mathcal{P}_2$, Theorem 6.10 (c) says that $\mathcal{B}$ is a basis for $\mathcal{P}_2$.

(23) Suppose that c_1, c_2, c_3 are scalars such that

$$c_1(1-x) + c_2(1-x^2) + c_3(x - x^2) = 0 + 0x + 0x^2.$$

Rearranging we see that this equation is equivalent to

$$(c_1 + c_2) + (-c_1 + c_3)x + (-c_2 - c_3)x^2 = 0 + 0x + 0x^2.$$

Comparing coefficients this gives the linear system

$$\begin{aligned} c_1 + c_2 &= 0 \\ -c_1 + c_3 &= 0 \\ -c_2 - c_3 &= 0 \end{aligned}$$

We form the augmented matrix and row-reduce:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since there are free variables, we see that the polynomials in $\mathcal{B}$ are not linearly independent. In fact, if we solve the system for c_1, c_2, c_3, we see that

$$x - x^2 = -(1-x) + (1-x^2).$$

Therefore, $\mathcal{B}$ is not a basis for $\mathcal{P}_2$.

(27) We want to find scalars c_1, c_2, c_3 and c_4 such that

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Comparing corresponding entries, this gives the linear system:

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 1 \\ c_2 + c_3 + c_4 &= 2 \\ c_3 + c_4 &= 3 \\ c_4 = 4 \end{aligned}$$

We form the augmented matrix and row-reduce:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

We conclude that $c_1 = c_2 = c_3 = -1$ and $c_4 = 4$. Thus,

$$[A]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}.$$

(29) We want to find scalars c_1, c_2 and c_3 such that

$$c_1(1) + c_2(1+x) + c_3(-1+x^2) = 2 - x + 3x^2.$$

Rearranging this is equivalent to the equation

$$(c_1 + c_2 - c_3) + c_2 x + c_3 x^2 = 2 - x + 3x^2.$$

Comparing coefficients, we obtain the linear system

$$\begin{aligned} c_1 + c_2 - c_3 &= 2 \\ c_2 &= -1 \\ c_3 &= 3 \end{aligned}$$

This has the solution $c_1 = 6, c_2 = -1$ and $c_3 = 3$. Therefore,

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix}.$$

(35) Note that

$$\begin{aligned} V &= \{p(x) \in \mathcal{P}_2 : p(1) = 0\} \\ &= \{p(x) = a + bx + cx^2 \in \mathcal{P}_2 : p(1) = 0\} \\ &= \{p(x) = a + bx + cx^2 \in \mathcal{P}_2 : a + b + c = 0\} \\ &= \{p(x) = a + bx + (-a - b)x^2 \in \mathcal{P}_2\} \\ &= \{p(x) = a(1 - x^2) + b(x - x^2) \in \mathcal{P}_2\} \\ &= span(1 - x^2, x - x^2) \end{aligned}$$

We now verify that our spanning polynomials are also linearly independent. So, suppose we have scalars c_1 and c_2 such that

$$c_1(1 - x^2) + c_2(x - x^2) = 0 + 0x + 0x^2.$$

Equivalently,

$$c_1 + c_2x + (-c_1 - c_2)x^2 = 0 + 0x + 0x^2.$$

Comparing coefficients, we see that $c_1 = c_2 = 0$. Thus,

$$\mathcal{B} = \{1 - x^2, x - x^2\}$$

is a basis for V. Thus, $\dim(V) = 2$.

(39) Note that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in V if and only if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which is true if and only if

$$\begin{bmatrix} a & (a+b) \\ c & (c+d) \end{bmatrix} = \begin{bmatrix} (a+c) & (b+d) \\ c & d \end{bmatrix}.$$

Comparing entries of the matrices, we must have $c = 0$ and $a = d$. So,

$$\begin{aligned} V &= \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, d \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\} \\ &= span \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \end{aligned}$$

We now verify that the above two matrices that span V are also linearly independent. Suppose we have scalars c_1 and c_2 such that

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Comparing entries we see that $c_1 = c_2 = 0$. Thus,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

is a basis for V. Thus, $\dim(V) = 2$.

(45) We need to find a polynomial $f(x)$ in $\mathcal{P}_2$ which makes the set $\{1+x, 1+x+x^2, f(x)\}$ linearly independent. Let $f(x) = 1$. If c_1, c_2, c_3 are scalars such that

$$c_1(1+x) + c_2(1+x+x^2) + c_3(1) = 0 = 0x + ox^2$$

then

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 &= 0 \end{aligned}$$

This system has only the trivial solution $c_1 = c_2 = c_3 = 0$. Thus, the set $\mathcal{B} = \{1+x, 1+x+x^2, 1\}$ is linearly independent. Since $\dim(\mathcal{P}_2) = 3$ and $\mathcal{B}$ consists of three linearly independent polynomials in $\mathcal{P}_2$, Theorem 6.10 (c) implies that $\mathcal{B}$ is a basis for $\mathcal{P}_2$.

(51) Since we already have a spanning set for the subspace, we simply need to throw away the polynomials which depend on the others. The first two polynomials are not scalar multiples of one another and so $\{1-x, x-x^2\}$ is a linearly independent set.

Observe that the third and fourth polynomials are linear combinations of the first two, i.e.,

$$\begin{aligned} 1-x^2 &= (1-x)+(x-x^2) \\ 1-2x+x^2 &= (1-x)-(x-x^2) \end{aligned}$$

and so we throw away the third and fourth given polynomials.

Therefore, $\{1-x, x-x^2\}$ is a linearly independent set of vectors that spans the given subspace. We conclude that $\{1-x, x-x^2\}$ is a basis for the subspace in question.

(53) Since we already have a spanning set for the subspace, we simply need to throw away the functions which depend on the others. The first two functions are not scalar multiples of one another and so $\{\sin^2 x, \cos^2 x\}$ is a linearly independent set.

Observe that the third function is a linear combination of the first two via the trig identity

$$\cos 2x = \cos^2 x - \sin^2 x$$

and so we throw away the third function.

Therefore, $\{\sin^2 x, \cos^2 x\}$ is a linearly independent set of vectors that spans the given subspace. We conclude that $\{\sin^2 x, \cos^2 x\}$ is a basis for the subspace in question.

Section 6.3:

(2) (a) Observe that

$$\mathbf{x} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus,

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

Similarly,

$$\mathbf{x} = -7\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Thus,

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}.$$

(b) We need to find the coordinate vector of each basis vector in $\mathcal{B}$ with respect to $\mathcal{C}$. We have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{3}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Thus, by definition,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -3/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

(c) We have

$$P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -3/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix} = [\mathbf{x}]_{\mathcal{C}}.$$

(d) We need to find the coordinate vector of each basis vector in $\mathcal{C}$ with respect to $\mathcal{B}$. We have

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus, by definition,

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}.$$

(e) We have

$$P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -7 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}}.$$

(3) (a) Observe that

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Similarly,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

(b) We need to find the coordinate vector of each basis vector in $\mathcal{B}$ with respect to $\mathcal{C}$. We have

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, by definition,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

(c) We have

$$P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = [\mathbf{x}]_{\mathcal{C}}.$$

(d) We need to find the coordinate vector of each basis vector in $\mathcal{C}$ with respect to $\mathcal{B}$. We have

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, by definition,

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(e) We have

$$P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}}.$$

(9) (a) We have

$$A = 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus,

$$[A]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix}.$$

To find $[A]_{\mathcal{C}}$ we need to find scalars such that

$$A = c_1 \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This leads to the system

$$\begin{aligned} c_1 + 2c_2 + c_3 + c_4 &= 4 \\ 2c_1 + c_2 + c_3 &= 2 \\ c_2 &= 0 \\ -c_1 + c_3 + c_4 &= -1 \end{aligned}$$

This system has solution $c_1 = 5/2, c_2 = 0, c_3 = -3$ and $c_4 = 9/2$. Thus

$$[A]_{\mathcal{C}} = \begin{bmatrix} 5/2 \\ 0 \\ -3 \\ 9/2 \end{bmatrix}.$$

(b) We need to find the coordinate vector of each basis vector in $\mathcal{B}$ with respect to $\mathcal{C}$. To do so we will need to solve 4 systems of equations similar to those in part (a). We omit the details of the systems here but give the linear combinations that arise. We have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + 0 \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} - 1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + 0 \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = - \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + 0 \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, by definition,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 0 & -1 & -1/2 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 3/2 & -1 & -2 & -1/2 \end{bmatrix}.$$

(c) We have

$$P_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix} 1/2 & 0 & -1 & -1/2 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 3/2 & -1 & -2 & -1/2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 0 \\ -3 \\ 9/2 \end{bmatrix} = [A]_{\mathcal{C}}.$$

(d) We need to find the coordinate vector of each basis vector in $\mathcal{C}$ with respect to $\mathcal{B}$. We have

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, by definition,

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}.$$

(e) We have

$$P_{\mathcal{B}\leftarrow\mathcal{C}}[A]_{\mathcal{C}} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5/2 \\ 0 \\ -3 \\ 9/2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix} = [A]_{\mathcal{B}}.$$

(11) (a) Observe that

$$f(x) = 2(\sin x + \cos x) - 5\cos x.$$

Thus,

$$[f(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

Similarly,

$$f(x) = 2(\sin x + \cos x) - 3\cos x.$$

Thus,

$$[f(x)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

(b) We need to find the coordinate vector of each basis vector in $\mathcal{B}$ with respect to $\mathcal{C}$. We have

$$\sin x + \cos x = 1(\sin x) + 1(\cos x)$$

and

$$\cos x = 0(\sin x) + 1(\cos x).$$

Thus, by definition,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(c) We have

$$P_{\mathcal{C}\leftarrow\mathcal{B}}[f(x)]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} = [f(x)]_{\mathcal{C}}.$$

(d) We need to find the coordinate vector of each basis vector in $\mathcal{C}$ with respect to $\mathcal{B}$. We have

$$\sin x = 1(\sin x + \cos x) - (\cos x)$$

and

$$\cos x = 0(\sin x + \cos x) + \cos x.$$

Thus, by definition,

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

(e) We have

$$P_{\mathcal{B}\leftarrow\mathcal{C}}[f(x)]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} = [f(x)]_{\mathcal{B}}.$$

(13) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Using Example 3.58, we see that the new $x'y'$-axes for the plane are

$$\mathcal{C} = \left\{ \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}, \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} \right\}.$$

To find the $x'y'$-coordinates, we need to find the matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$. We see that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} - \frac{\sqrt{3}}{2} \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\sqrt{3}}{2} \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}.$$

Thus,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}.$$

(a) We calculate

$$\left[\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 + \sqrt{3} \\ (-3\sqrt{3})/2 + 1 \end{bmatrix}$$

(b) Let $\mathbf{x}$ be the vector whose $x'y'$-coordinates are $(4, -4)$. We want to find $[\mathbf{x}]_{\mathcal{B}}$. We calculate

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}\,[\mathbf{x}]_{\mathcal{C}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 + 2\sqrt{3} \\ 2\sqrt{3} - 2 \end{bmatrix}$$

(14) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Using Example 3.58, we see that the new $x'y'$-axes for the plane are

$$\mathcal{C} = \left\{ \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} \right\}.$$

To find the $x'y'$-coordinates, we need to find the matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$. We see that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{\sqrt{2}} \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}.$$

Thus,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

(a) We calculate

$$\left[\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -5/\sqrt{2} \end{bmatrix}$$

(b) Let $\mathbf{x}$ be the vector whose $x'y'$-coordinates are $(4, -4)$. We want to find $[\mathbf{x}]_{\mathcal{B}}$.

We calculate

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}\, [\mathbf{x}]_{\mathcal{C}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8/\sqrt{2} \end{bmatrix}$$