Problem Set 3 Solutions

Due: Thursday, April 15

This problem set involves choices! Submit solutions to 2 exercises from Part I and 1 exercise from Part II.

Part I - Exercises Related to Hilbert Functions & Regular Sequences

(1) For parts (b) - (d) of this exercise use reverse-lexicographic order with $x_1 >_{revlex}> x_2 >_{revlex} \cdots$.

(a) Find a $(3,4,5)$-lex-plus-powers ideal $L \subset S = k[x_1, x_2, x_3]$ such that $H(S/L, 3) = 9$ and $H(S/L, 6) = 5$.

Solution: Let $L = (x_1^3, x_2^4, x_3^5, x_1^2x_2^2, x_1^2x_2x_3, x_1^2x_3^2)$. A straightforward check using the definition verifies that L is a $(3,4,5)$-lex-plus-powers ideal. One can also check, either by hand or using a computer algebra program, that $H(S/L) = (1,3,6,9,8,7,5,3,1,0,0,\ldots)$. Another possible $(3,4,5)$-lex-plus-powers ideal that would satisfy the given conditions is $L' := (x_1^3, x_2^4, x_3^5, x_1^2x_3^4, x_1^2x_2x_3^3, x_1^2x_2^2x_3^2, x_1^2x_2^3x_3)$. We have $H(S/L') = (1,3,6,9,11,11,5,3,1,0,0,\ldots)$.

(b) Fix m to be a monomial of degree d in $S = k[x_1,x_2,x_3,x_4]/(x_1^5, x_2^4, x_3^4, x_4^3)$. Recall that $L(m)$ denotes the set of all degree d monomials in S which are greater than or equal to m. Decompose $|L(x_1^3x_2^3x_4^2)|$ in terms of integers of the form $\binom{e_1,\ldots,e_j}{l}$. Give an algebraic description of each term in the decomposition.

Solution: The desired decomposition is

$$|L(x_1^3x_2^3x_4^2)| = \binom{4,3,3}{8} + \binom{4,3,3}{7} + \binom{4,3}{6} = 6 + 10 + 2 = 18.$$

The integers in this decomposition are counting monomials in S of degree 8 as follows:

Any degree 8 monomial in S not divisible by x_4 will be greater than $x_1^3x_2^3x_4^2$. There are

$$6 = \binom{4,3,3}{8}$$

such monomials; namely, those of the form $x_1^{a_1}x_2^{a_2}x_3^{a_3}$ where $0 \le a_1 \le 4, 0 \le a_2 \le 3, 0 \le a_3 \le 3$ and $a_1 + a_2 + a_3 = 8$.

Any degree 8 monomial in S involving x_4^1 but no higher power of x_4 will be greater than $x_1^3x_2^3x_4^2$. There are

$$10 = \binom{4,3,3}{7}$$

such monomials; namely, those of the form $x_1^{a_1}x_2^{a_2}x_3^{a_3}$ where $0 \le a_1 \le 4, 0 \le a_2 \le 3, 0 \le a_3 \le 3$ and $a_1 + a_2 + a_3 = 7$.

The monomials of degree 8 in S involving x_4^2 but no higher power of x_4 and which are greater than $x_1^3x_2^3x_4^2$ are $x_1^4x_2^2x_4^2$ and $x_1^3x_2^3x_4^2$. This set of monomials has cardinality

$$2 = \binom{4,3}{6}.$$

(c) Assume $I \subset S = k[x_1,x_2,x_3,x_4]$ is a homogeneous ideal containing $\{x_1^5, x_2^4, x_3^4, x_4^3\}$. If $H(S/I, 8) = 17$, then what is the largest value possible for $H(S/I, 9)$?

Solution: To answer this question we need to use the following "Pascal's Table" associated to the given powers of the variables:

Degree:	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$H(k[x_1]/(x_1^5))$:	1	1	1	1	**1**	$\underline{\mathbf{0}}$	$\underline{\mathbf{0}}$	$\underline{0}$	0	0	0	0	0	0
$H(k[x_1,x_2]/(x_1^5,x_2^4))$:	1	2	3	4	4	3	2	1	0	0	0	0	0	0
$H(k[x_1,x_2,x_3]/(x_1^5,x_2^4,x_3^4))$:	1	3	6	10	13	14	13	**10**	$\underline{\mathbf{6}}$	$\underline{3}$	1	0	0	0
$H(k[x_1,x_2,x_3,x_4]/(x_1^5,x_2^4,x_3^4,x_4^3))$:	1	4	10	19	29	37	40	37	29	19	10	4	1	0

Starting in degree 8, we decompose 17 using integers of the form $\binom{e_1,\ldots,e_j}{l}$. Doing so we obtain the decomposition (the corresponding numbers in the table are in bold):

$$H(S/I,8) = 17 = \binom{4,3,3}{8} + \binom{4,3,3}{7} + \binom{4}{6} + \binom{4}{5} + \binom{4}{4} = 6 + 10 + 0 + 0 + 1.$$

We saw in class that

$$H(S/I,9) \leq \binom{4,3,3}{9} + \binom{4,3,3}{8} + \binom{4}{7} + \binom{4}{6} + \binom{4}{5} = 3 + 6 + 0 + 0 + 0 = 9$$

(the corresponding numbers in the table are underlined).

(d) Assume that the EGH Conjecture is true. Can there be a homogeneous $(3,4,4,5)$-ideal $I \subset S = k[x_1,x_2,x_3,x_4]$ with $H(S/I) = (1,4,10,18,24,29,\ldots)$?

Solution: We again use the "Pascal's Table":

Degree:	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$H(k[x_1]/(x_1^5))$:	1	1	**1**	$\underline{1}$	1	0	0	0	0	0	0	0	0	0
$H(k[x_1,x_2]/(x_1^5,x_2^4))$:	1	2	3	4	4	3	2	1	0	0	0	0	0	0
$H(k[x_1,x_2,x_3]/(x_1^5,x_2^4,x_3^4))$:	1	3	6	**10**	**$\underline{13}$**	$\underline{14}$	13	10	6	3	1	0	0	0
$H(k[x_1,x_2,x_3,x_4]/(x_1^5,x_2^4,x_3^4,x_4^3))$:	1	4	10	19	29	37	40	37	29	19	10	4	1	0

Let $\mathcal{H} = (1,4,10,18,24,29,\ldots) = \{h_t\}_{t\geq 0}$. Note that we can decompose $h_4 = 24$ as:

$$h_4 = 24 = \binom{4,3,3}{4} + \binom{4,3,3}{3} + \binom{4}{2} = 13 + 10 + 1$$

(the corresponding integers are in bold in the table). Assuming the EGH Conjecture is true, if there were a homogeneous $(3,4,4,5)$-ideal $I \subset S = k[x_1,x_2,x_3,x_4]$ with $H(S/I) = (1,4,10,18,24,29,\ldots) = \mathcal{H}$ then

$$h_5 = 29 \leq \binom{4,3,3}{5} + \binom{4,3,3}{4} + \binom{4}{3}$$

(these corresponding integers are in underlines in the table). Since

$$h_5 = 29 > 28 = 14 + 13 + 1 = \binom{4,3,3}{5} + \binom{4,3,3}{4} + \binom{4}{3},$$

we conclude that there can be no such ideal I.

(2) *EGH Points Conjecture in* $\mathbb{P}^2$: Fix integers $2 \leq d_1 \leq d_2$. Let $\Delta\mathcal{H} = \{h_t\}_{t\geq 0}$ be the first difference Hilbert function of some finite set of distinct points in $\mathbb{P}^2$ such that $h_t \leq H(k[x_1,x_2]/(x_1^{d_1},x_2^{d_2}),t)$ for all $t \geq 0$. Prove that there exist finite sets of distinct points $\mathbb{X} \subseteq \mathbb{Y} \subset \mathbb{P}^2$ where $\mathbb{Y}$ is a complete intersection of type $\{d_1,d_2\}$ and $\Delta H(\mathbb{X}) = \Delta\mathcal{H}$ if and only if $h_{t+1} \leq h_t^{(t)}$ for all $t \geq 1$.

Proof. No-one submitted a solution for this exercise. However, a few people have indicated that they are still thinking about a proof. Thus, rather than giving an entire proof I will provide only hints.

Suppose that $h_{t+1} \leq h_t^{(t)}$ for all $t \geq 1$. To construct the sets $\mathbb{X}$ and $\mathbb{Y}$, define

$$\mathbb{Y} := \{[1 : a_1 : a_2] \mid a_i \in \mathbb{N}, 0 \leq a_1 \leq d_2 - 1, 0 \leq a_2 \leq d_1 - 1\}.$$

By carefully applying the work of Clements and Lindström, one can lift a certain monomial ideal in $k[x_1,x_2]$ to obtain the desired subset $\mathbb{X} \subseteq \mathbb{Y}$. (Remember: the bounds $h_{t+1} \leq h_t^{(t)}$ for all $t \geq 1$ really do come from the work of Clements and Lindström.)

Now suppose that we have sets $\mathbb{X} \subseteq \mathbb{Y} \subset \mathbb{P}^2$ where $\mathbb{Y}$ is a complete intersection of type $\{d_1,d_2\}$ and $\Delta H(\mathbb{X}) = \Delta\mathcal{H}$. To show that $h_{t+1} \leq h_t^{(t)}$ for all $t \geq 1$, combine the following observations:

- If $t \leq d_2 - 2$ and $h_t < H(k[x_1,x_2]/(x_1^{d_1}, x_2^{d_2}), t)$, then $h_t^{(t)} = h_t$.
- If $t \geq d_2 - 1$ and $h_t < H(k[x_1,x_2]/(x_1^{d_1}, x_2^{d_2}), t)$, then $h_t^{(t)} = h_t - 1$.
- $\Delta\mathcal{H}$ must be an O-sequence.
- One cannot have $h_t = h_{t+1}$ for any $t \in \{d_2 - 1, \ldots, d_1 + d_2 - 3\}$. (You can show this last fact by using contradiction and applying the Cayley-Bacharach Theorem.)

□

(3) *Classical Cayley-Bacharach Theorem*: Let $\mathbb{X} = \{P_1, \ldots, P_9\}$ be the complete intersection of two cubics in $\mathbb{P}^2$. Use the Cayley-Bacharach Theorem to show that any cubic passing through 8 of the 9 points of $\mathbb{X}$ must also pass through the remaining 9th point.

Proof. Without loss of generality, we can assume that there is a cubic passing through $\mathbb{Y} := \{P_1, \ldots, P_8\}$. We want to show that this cubic also passes through $\{P_9\}$. Since $\mathbb{X}$ is a complete intersection of two cubics, we know that

$$\Delta H(\mathbb{X}) = (1, 2, 3, 2, 1, 0, 0, \ldots).$$

Also, by properties of Hilbert functions of finite sets of distinct points, we know that

$$\Delta H(\{P_9\}) = (1, 0, 0, \ldots).$$

The Cayley-Bacharach Theorem gives the relationship

$$\Delta H(\mathbb{X}, t) = \Delta H(\{P_9\}, t) + \Delta H(\mathbb{Y}, (3 + 3) - 2 - t).$$

Using this equation to solve for $\Delta H(\mathbb{Y})$, we find

$$\Delta H(\mathbb{Y}) = (1, 2, 3, 2, 0, 0, \ldots).$$

Thus, we now have

$$H(\mathbb{X}) = H(\{P_1, \ldots, P_8, P_9\}) = (1, 3, 6, 8, 9, 9, \ldots)$$
$$H(\mathbb{Y}) = H(\{P_1, \ldots, P_8\}) = (1, 3, 6, 8, 8, \ldots).$$

That is, $\dim_k(I(\mathbb{Y})_3) = \dim_k(I(\mathbb{X})_3) = \binom{2+3}{3} - 8 = 10 - 8 = 2$. But, since $\mathbb{Y} \subset \mathbb{X}$, we know that $I(\mathbb{X})_3 \subseteq I(\mathbb{Y})_3$. Thus, $I(\mathbb{X})_3 = I(\mathbb{Y})_3$. That is, any cubic passing through $\mathbb{Y}$ must pass through all of $\mathbb{X}$ and hence $\{P_9\}$. □

Part II - Exercises From Group Presentations

(1) *From Croll-Gibbons-Johnson*: Our exercise outlines a proof of the following lemma due to Buchsbaum and Eisenbud:

Lemma. *Let R be a ring, $x \in R$, and $S = R/(x)$. Let B be an S-module, and let*

$$\mathcal{F}: \qquad F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

be an exact sequence of S-modules with $\operatorname{coker}(\phi_1) \cong B$. *Suppose that*

$$\mathcal{G}: \qquad G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$$

is a complex of R-modules such that
(i) x is a non-zero divisor on each G_i,
(ii) $G_i \otimes_R S \cong F_i$, and
(iii) $\psi_i \otimes_R S = \phi_i$.
Then $A =$ $\operatorname{coker}(\psi_1)$ *is a lifting of B to R.*

(a) With the conditions of the lemma and $i \in \{0, 1, 2\}$, prove that the sequence

$$0 \longrightarrow G_i \xrightarrow{\cdot x} G_i \xrightarrow{q} G_i/xG_i \longrightarrow 0$$

is exact, where $\cdot x$ is the map given by multiplication by x and q is the canonical quotient map.

Proof. The map $\cdot x$ is injective since x is a non-zero divisor on G_i (condition (i)). By construction, $\ker(q) = xG_i = \operatorname{im}(\cdot x)$. Finally, the quotient is surjective. $\square$

(b) In the diagram below, show that each square of the diagram commutes.

$$
\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & G_2 & \xrightarrow{\psi_2} & G_1 & \xrightarrow{\psi_1} & G_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & \downarrow{\scriptstyle \cdot x} & & \downarrow{\scriptstyle \cdot x} & & \downarrow{\scriptstyle \cdot x} & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & G_2 & \xrightarrow{\psi_2} & G_1 & \xrightarrow{\psi_1} & G_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & F_2 & \xrightarrow{\phi_2} & F_1 & \xrightarrow{\phi_1} & F_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & 0 & & & &
\end{array}
$$

Conclude that

$$0 \longrightarrow \mathcal{G} \xrightarrow{\cdot x} \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow 0$$

is an exact sequence of complexes (briefly explain why each column is exact).

Proof. Since $x \in R$ and the ψ_i are module homomorphisms, $\cdot x \circ \psi_i(g) = x\psi_i(g) = \psi_i(xg) = \psi_i \circ \cdot x(g)$, so the top squares commute. Note that $F_i \cong G_i \otimes S \cong G_i/xG_i$ (which gives that the columns are exact), and we may rewrite the square as

$$
\begin{array}{ccc}
G_i & \xrightarrow{\quad\psi_i\quad} & G_{i-1} \\
{\scriptstyle \mathrm{id}_{G_i} \otimes 1_S}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{G_{i-1}} \otimes 1_S} \\
G_i \otimes_R S & \xrightarrow[\psi_i \otimes \mathrm{id}_S]{} & G_{i-1} \otimes_R S.
\end{array}
$$

But then

$$\left(\mathrm{id}_{G_{i-1}} \otimes 1_S\right) \circ \psi_i(g) = \psi_i(g) \otimes 1_S = (\psi_i \otimes \mathrm{id}_S) \circ (\mathrm{id}_{G_i} \otimes 1_S)(g),$$

and the square commutes. $\square$

(c) Given any exact sequence of complexes $0 \longrightarrow D_. \xrightarrow{\cdot x} D_. \longrightarrow C_. \longrightarrow 0$, there is a corresponding long exact sequence in homology given by

$$
\begin{array}{ccccc}
\cdots \longrightarrow & H_2(D_.) & \longrightarrow & H_2(C_.) & \\
 & & \swarrow & & \\
H_1(D_.) \xrightarrow{\cdot x} & H_1(D_.) & \longrightarrow & H_1(C_.) & \\
 & & \swarrow & & \\
H_0(D_.) \xrightarrow{\cdot x} & H_0(D_.) & \longrightarrow & H_0(C_.) & \longrightarrow 0.
\end{array}
$$

Use the long exact sequence in homology with the exact sequence of complexes to determine that $A/xA \cong B$ and x is a non-zero divisor on A. Conclude that A is a lifting of B to R.

Proof. Computing homologies, we determine that $H_0(\mathcal{F}) = \operatorname{coker}(\phi_1) = B$, $H_0(\mathcal{G}) = \operatorname{coker}(\psi_1) = A$, and $H_1(\mathcal{F}) = 0$ since $\mathcal{F}$ is exact at F_1. Thus the long exact sequence in homology yields, in an exciting role reversal, a short exact sequence

$$0 \longrightarrow A \xrightarrow{\cdot x} A \longrightarrow B \longrightarrow 0,$$

confirming that $\cdot x$ is injective (so x is a non-zero divisor on A) and $B \cong A/\operatorname{im}(\cdot x) = A/xA$. That's what we needed to satisfy to show that A is a lifting of B to R. □

(2) *From Brase-Denkert-Janssen*: Accept that any monomial ordering $>$ on $k[x_1, \dots, x_n]$ can be obtained by taking pairwise orthogonal vectors $\mathbf{v_1}, \dots, \mathbf{v_r} \in k^n$ where $\mathbf{v_1}$ has only non-negative entries and where $\mathbf{x}^{\boldsymbol{\alpha}} > \mathbf{x}^{\boldsymbol{\beta}}$ if and only if there exists $t \leq r$ such that $\mathbf{v_i} \cdot \boldsymbol{\alpha} = \mathbf{v_i} \cdot \boldsymbol{\beta}$ for all $i \leq t-1$ and $\mathbf{v_t} \cdot \boldsymbol{\alpha} > \mathbf{v_t} \cdot \boldsymbol{\beta}$.

(a) Let $r = n$ and $\mathbf{v_i} = \mathbf{e_i}$ for all i where $\mathbf{e_i}$ is the ith standard basis vector for k^n. Show that $>$ is the lexicographic order.

Proof. Let $\boldsymbol{\alpha} = (a_1, \dots, a_n)$ and $\boldsymbol{\beta} = (b_1, \dots, b_n)$. By definition, $\mathbf{x}^{\boldsymbol{\alpha}} >_{lex} \mathbf{x}^{\boldsymbol{\beta}}$ if and only if the leftmost non-zero entry of $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is positive. Say this entry is in the t position. Then $\mathbf{x}^{\boldsymbol{\alpha}} >_{lex} \mathbf{x}^{\boldsymbol{\beta}}$ if and only if

$$\mathbf{v_i} \cdot \boldsymbol{\alpha} = a_i = b_i = \mathbf{v_i} \cdot \boldsymbol{\beta} \text{ for all } 1 \leq i \leq t-1$$

and

$$\mathbf{v_t} \cdot \boldsymbol{\alpha} = a_t > b_t = \mathbf{v_t} \cdot \boldsymbol{\beta}.$$

Therefore, $\mathbf{x}^{\boldsymbol{\alpha}} >_{lex} \mathbf{x}^{\boldsymbol{\beta}}$ if and only if $\mathbf{x}^{\boldsymbol{\alpha}} > \mathbf{x}^{\boldsymbol{\beta}}$. □

(b) Let $r = n$ and define vectors as follows:

$$\mathbf{v_1} = (1, \dots, 1)$$
$$\mathbf{v_i} = (1, 1, \dots, 1, i-(n+1), 0, 0, \dots, 0)$$

where the entry $i-(n+1)$ is in the $(n+2-i)$th position for $i \in \{2, \dots, n\}$. Show that $>$ is the graded reverse-lexicographic order.

Proof. Let $r = n$ and define vectors $\mathbf{v_1}$ and $\mathbf{v_i}$ as above. We will show that $>$ is $>_{\text{grevlex}}$. To do this, we will show that $\boldsymbol{x}^{\boldsymbol{\alpha}} > \boldsymbol{x}^{\boldsymbol{\beta}}$ iff there exists $t \leq n$ such that $\mathbf{v_i} \cdot \boldsymbol{\alpha} = \mathbf{v_i} \cdot \boldsymbol{\beta}$ for $i \leq t-1$ and $\mathbf{v_t} \cdot \boldsymbol{\alpha} > \mathbf{v_{t'}} \cdot \boldsymbol{\beta}$.

Let $\boldsymbol{\alpha} = (a_1, \dots, a_n)$ and $\boldsymbol{\beta} = (b_1, \dots, b_n)$ we have $\mathbf{v_1} \cdot \boldsymbol{\alpha} = |\boldsymbol{\alpha}| = \sum_{i=1}^n a_i$ and likewise for $\boldsymbol{\beta}$. Additionally, $\mathbf{v_2} \cdot \boldsymbol{\alpha} = \sum_{i=1}^{n-1} a_i - n a_n$ and likewise for $\boldsymbol{\beta}$. Inductively, we can see that

(1) $$\mathbf{v_i} \cdot \boldsymbol{\alpha} = \mathbf{v_{i-1}} \cdot \boldsymbol{\alpha} + (n+2-i)(a_{n-(i-3)} - a_{n-(i-2)}) \qquad \text{for } i \geq 3$$

and likewise for $\boldsymbol{\beta}$.

Suppose $\boldsymbol{x}^{\boldsymbol{\alpha}} >_{\text{grevlex}} \boldsymbol{x}^{\boldsymbol{\beta}}$ for $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. By definition, this happens IFF $\mathbf{v_1} \cdot \boldsymbol{\alpha} > \mathbf{v_1} \cdot \boldsymbol{\beta}$ or $\mathbf{v_1} \cdot \boldsymbol{\alpha} = \mathbf{v_1} \cdot \boldsymbol{\beta}$ and there exists s for which $0 \leq s < n$ such that $a_{n-i} = b_{n-i}$ for all $i < s$ and $a_{n-s} < b_{n-s}$. We will show that this is true IFF $\boldsymbol{x}^{\boldsymbol{\alpha}} > \boldsymbol{x}^{\boldsymbol{\beta}}$. To do so, we consider two cases.

Case 1: Assume $|\boldsymbol{\alpha}| > |\boldsymbol{\beta}|$. This holds IFF $\mathbf{v_1} \cdot \boldsymbol{\alpha} > \mathbf{v_1} \cdot \boldsymbol{\beta}$ which implies $\boldsymbol{x}^{\boldsymbol{\alpha}} > \boldsymbol{x}^{\boldsymbol{\beta}}$ and concludes Case 1.

Case 2: Assume $\mathbf{v_1} \cdot \boldsymbol{\alpha} = \mathbf{v_1} \cdot \boldsymbol{\beta}$ and there exists $s < n$ such that $a_{n-i} = b_{n-i}$ for all $i < s$ and $a_{n-s} < b_{n-s}$. Now $s = 0$ IFF $\mathbf{v_2} \cdot \boldsymbol{\alpha} > \mathbf{v_2} \cdot \boldsymbol{\beta}$, as this holds IFF $-na_n > -nb_n$ IFF $a_n < b_n$.

So, assume $s > 0$. Since $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ but $|\boldsymbol{\alpha}| = \sum_{i=1}^n a_i = \sum_{i=1}^n b_i = |\boldsymbol{\beta}|$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ must differ in at least two entries, so $a_{n-i} = b_{n-i}$ for $i < s$ means that $s \leq n-2$. Thus, we are considering s for which $1 \leq s \leq n-2$.

Notice that $1 \leq s \leq n-2$ implies that $\mathbf{v_1} \cdot \boldsymbol{\alpha} = \mathbf{v_1} \cdot \boldsymbol{\beta}$ and $\mathbf{v_2} \cdot \boldsymbol{\alpha} = \mathbf{v_2} \cdot \boldsymbol{\beta}$. Using (1), we see that $\mathbf{v_3} \cdot \boldsymbol{\alpha} = \mathbf{v_2} \cdot \boldsymbol{\alpha} + (n-1)(a_n - a_{n-1})$ (and likewise for $\boldsymbol{\beta}$), so $\mathbf{v_3} \cdot \boldsymbol{\alpha} - \mathbf{v_3} \cdot \boldsymbol{\beta} = (n-1)(b_{n-1} - a_{n-1})$, so $\mathbf{v_3} \cdot \boldsymbol{\alpha} > \mathbf{v_3} \cdot \boldsymbol{\beta}$ IFF $a_{n-1} < b_{n-1}$ and $\mathbf{v_3} \cdot \boldsymbol{\alpha} = \mathbf{v_3} \cdot \boldsymbol{\beta}$ IFF $a_{n-1} = b_{n-1}$.

Define $t = s + 2$. Continuing like this, we see for all $t > i \geq 3$ that

$$\mathbf{v_i} \cdot \boldsymbol{\alpha} - \mathbf{v_i} \cdot \boldsymbol{\beta} = (n + 2 - i)(b_{n-(i-2)} - a_{n-(i-2)}).$$

Thus, for all such i, $\mathbf{v_i} \cdot \boldsymbol{\alpha} = \mathbf{v_i} \cdot \boldsymbol{\beta}$. Now, if $i = s + 2 = t$,

$$\mathbf{v_t} \cdot \boldsymbol{\alpha} - \mathbf{v_t} \cdot \boldsymbol{\beta} = (n + 2 - (s + 2))(b_{n-(s+2-2)} - a_{n-(s+2-2)}) = (n - s)(b_{n-s} - a_{n-s}) > 0.$$

In other words, $a_{n-s} < b_{n-s}$ IFF $\mathbf{v_t} \cdot \boldsymbol{\alpha} > \mathbf{v_t} \cdot \boldsymbol{\beta}$. Thus, $a_{n-s} < b_{n-s}$ implies $\boldsymbol{x}^{\boldsymbol{\alpha}} > \boldsymbol{x}^{\boldsymbol{\beta}}$.

Now, if $\boldsymbol{x}^{\boldsymbol{\alpha}} > \boldsymbol{x}^{\boldsymbol{\beta}}$, either the converse of the last conclusion of Case 1 is true (i.e., $\boldsymbol{x}^{\boldsymbol{\alpha}} > \boldsymbol{x}^{\boldsymbol{\beta}}$ implies $\mathbf{v_1} \cdot \boldsymbol{\alpha} > \mathbf{v_1} \cdot \boldsymbol{\beta}$), or that of Case 2 is true (i.e., $\boldsymbol{x}^{\boldsymbol{\alpha}} > \boldsymbol{x}^{\boldsymbol{\beta}}$ implies $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and $a_{n-s} < b_{n-s}$ for some $s \geq 0$), and, in either case, we may retrace the string of "if and only if"s from there to conclude that $\boldsymbol{x}^{\boldsymbol{\alpha}} >_{\text{grevlex}} \boldsymbol{x}^{\boldsymbol{\beta}}$. □