Problem Set 2 Solutions

Due: Thursday, March 25

(1) Fix $\mathcal{H} := (1, 4, 6, 9, 10, 13, 13, \ldots)$ and let $S := k[x_1, x_2, x_3, x_4]$ where k is a field. Does there exist a homogeneous ideal $I \subset S$ such that $H(S/I) = \mathcal{H}$? Provide two reasons for your answer: one using an O-sequence approach and one using an order ideal of monomials approach.

Solution: Denote the ith component of $\mathcal{H}$ by d_i for $i \geq 0$.

- *O-Sequence Approach:* In order for $\mathcal{H} = H(S/I)$ for some homogeneous ideal I we need $d_{i+1} \leq d_i^{<i>}$ for $i \geq 1$ and $d_0 = 1$. Consider $i = 4$. We calculate the 4-binomial expansion of $d_4 = 10$ to be:
$$10 = \binom{5}{4} + \binom{4}{3} + \binom{2}{2}.$$
Thus,
$$10^{<4>} = \binom{6}{5} + \binom{5}{4} + \binom{3}{3} = 6 + 5 + 1 = 12.$$
But $d_5 = 13 > 10^{<4>} = 12$ and so $\mathcal{H}$ is not an O-sequence. By Macaulay's Theorem we conclude that there is no homogeneous ideal $I \subset S$ such that $H(S/I) = \mathcal{H}$.
- *Order Ideal of Monomials Approach:* For what follows we use the degree reverse lexicographic ordering with $x_1 > x_2 > \cdots > x_n$. Let $\mathcal{M} = \cup_{t \geq 0} \mathcal{M}_t$, where $\mathcal{M}_t$ is the set of d_t largest monomials of degree t. Setting $t = 4$ and $t = 5$ we find
$$\mathcal{M}_4 = \{x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4, x_1^3x_3, x_1^2x_2x_3, x_1x_2^2x_3, x_2^3x_3, x_1^2x_3^2\}$$
and
$$\mathcal{M}_5 = \{x_1^5, x_1^4x_2, x_1^3x_2^2, x_1^2x_2^3, x_1x_2^4, x_2^5, x_1^4x_3, x_1^3x_2x_3, x_1^2x_2^2x_3, x_1x_2^3x_3, x_2^4x_3, x_1^3x_3^2, x_1^2x_2x_3^2\}.$$
Note that $x_1x_2x_3^2$ is of degree 4 and divides $x_1^2x_2x_3^2 \in \mathcal{M}_5$. However, $x_1x_2x_3^2 \notin \mathcal{M}_4$. We conclude that $\mathcal{M}$ is not an order ideal of monomials. So, by Macaulay's Theorem, there does not exist a homogeneous ideal $I \subset S$ with $H(S/I) = \mathcal{H}$.

(2) For this exercise we use the same notation that was set up in our discussion of lifting monomial ideals. Let $f = \mathbf{x}^{\boldsymbol{\alpha}} \in S = k[x_1, \ldots, x_n]$. Prove the following two facts:

(a) $\overline{f}(\overline{\boldsymbol{\beta}}) = 0$ if and only if $\boldsymbol{\alpha} \not\leq \boldsymbol{\beta}$;

Proof. Let $\boldsymbol{\alpha} = (a_1, \ldots, a_n)$ and $\boldsymbol{\beta} = (b_1, \ldots, b_n)$. Recall that
$$\overline{f} = \prod_{j=1}^{n} \prod_{i=0}^{a_j - 1} (x_j - t_{j,i}x_0)$$
and
$$\overline{\boldsymbol{\beta}} = [1 : t_{1,b_1} : t_{2,b_2} : \cdots : t_{n,b_n}].$$
Using the fact that the $t_{l,m}$ are chosen to be distinct for a fixed l, we see that
$$\begin{aligned}\overline{f}(\overline{\boldsymbol{\beta}}) = 0 &\iff t_{j,b_j} = t_{j,i} \text{ for some } 1 \leq j \leq n \text{ and some } 0 \leq i \leq a_j - 1 \\ &\iff b_j = i \text{ for some } 1 \leq j \leq n \text{ and some } 0 \leq i \leq a_j - 1 \\ &\iff b_j < a_j \text{ for some } 1 \leq j \leq n \\ &\iff \boldsymbol{\alpha} \not\leq \boldsymbol{\beta}.\end{aligned}$$

□

(b) $\overline{f}(\overline{\boldsymbol{\gamma}}) = 0$ for all $\boldsymbol{\gamma}$ with $\deg(\boldsymbol{\gamma}) \leq \deg(\boldsymbol{\alpha})$ (except for $\boldsymbol{\alpha}$ itself).

Proof. Let $\boldsymbol{\alpha} = (a_1, \ldots, a_n)$ and $\boldsymbol{\gamma} = (c_1, \ldots, c_n)$ where $\deg(\boldsymbol{\gamma}) = \deg(\boldsymbol{\alpha})$ and $\boldsymbol{\gamma} \neq \boldsymbol{\alpha}$. It is easy to see that if $\deg(\boldsymbol{\gamma}) < \deg(\boldsymbol{\alpha})$ then there is some j such that $c_j < a_j$. If $\deg(\boldsymbol{\alpha}) = \deg(\boldsymbol{\gamma})$ then again there must exist j such that $c_j < a_j$; otherwise $c_i \geq a_i$ for all i implying that $\boldsymbol{\alpha} = \boldsymbol{\gamma}$ since $\sum_{i=1}^n c_i = \sum_{i=1}^n a_i$, a contradiction to the assumption $\boldsymbol{\alpha} \neq \boldsymbol{\gamma}$. We conclude that $\boldsymbol{\alpha} \nleq \boldsymbol{\gamma}$. Applying part (a) gives $\overline{f}(\overline{\boldsymbol{\gamma}}) = 0$ as desired. □

(3) In this exercise we further explore Hilbert functions of distinct points in projective 2-space. Let $S = k[x_1, x_2]$, where k is an algebraically closed field of characteristic zero. Further, let $J \subset S$ be a homogeneous ideal such that $\sqrt{J} = (x_1, x_2)$. We set $\alpha(J)$ to be the least degree of a non-zero homogeneous polynomial in J.

(a) Set $B = S/J$. Prove that

$$H(B,t) = \begin{cases} t+1 & \text{for } t < \alpha(J) \\ \leq \alpha(J) & \text{for } t \geq \alpha(J). \end{cases}$$

Proof. First observe that

$$H(B,t) = \dim_k(S_t) - \dim_k(J_t) = (t+1) - \dim_k(J_t).$$

For $t < \alpha(J)$, we have that $J_t = 0$ and so $H(B,t) = t+1$. For the second part of the claim, note that there exists a non-zero element $F \in J$ such that $\deg(F) = \alpha(J)$. Hence, $\dim_k(J_{\alpha(J)}) \geq 1$ and so $H(B, \alpha(J)) \leq (\alpha(J)+1) - 1 = \alpha(J)$. Finally, fix $t > \alpha(J)$. We have that $(F) \subseteq J$ and so

$$\dim_k(S_{t-\alpha(J)}) = \dim_k((F)_t) \leq \dim_k(J_t).$$

Therefore,

$$\begin{aligned} H(B,t) &= \dim_k(S_t) - \dim_k(J_t) \\ &\leq \dim_k(S_t) - \dim_k((F)_t) \\ &= \dim_k(S_t) - \dim_k(S_{t-\alpha(J)}) \\ &= (t+1) - (t - \alpha(J) + 1) \\ &= \alpha(J). \end{aligned}$$

□

(b) Let $V \subset S_t$ be a non-zero subspace of S_t. Denote by S_1V the subspace of S_{t+1} generated by $\{Lv \mid L \in S_1 \text{ and } v \in V\}$. Prove that

$$\dim_k(S_1V) \geq (\dim_k V) + 1.$$

Proof. Let $F_1, \ldots, F_l$ be a basis for V. It is clear that $\{x_1F_1, \ldots, x_lF_l, x_2F_1, \ldots, x_2F_l\}$ spans S_1V and $\{x_1F_1, \ldots, x_1F_l\}$ is a linearly independent set. We will be done if we can show that x_2F_j is not in the span of $\{x_1F_1, \ldots, x_1F_l\}$ for some j.

For $1 \leq i \leq l$ write $F_i = x_1^q F_i'$, where q is chosen so that x_1 does not divide some F_j' ($q = 0$ is possible). We claim that x_2F_j is not in the span of $\{x_1F_1, \ldots, x_1F_l\}$. To see this, assume to the contrary that we can find constants $c_1, \ldots, c_l \in k$ such that

$$\sum_{i=1}^{l} c_i x_1 F_i = x_2 F_j.$$

Then

$$x_1^{q+1}\sum_{i=1}^{l} c_i F_i' = x_2 x_1^q F_j'$$

which implies that x_1 divides F_j', a contradiction. □

(c) Let $\mathbb{X} = \{P_1, \ldots, P_t\}$ be a set of distinct points in $\mathbb{P}^2$. We set $\alpha = \alpha(\mathbb{X})$ to be the least degree of a non-zero homogeneous polynomial in $I(\mathbb{X})$. Show that $\Delta H(\mathbb{X})$ has the form

$$\Delta H(\mathbb{X}) = \{1, 2, 3, \ldots, \alpha - 1, \alpha, \Delta H(\mathbb{X}, \alpha), \Delta H(\mathbb{X}, \alpha + 1), \ldots\}$$

where $\alpha \geq \Delta H(\mathbb{X}, \alpha) \geq \Delta H(\mathbb{X}, \alpha + 1) \geq \Delta H(\mathbb{X}, \alpha + 2) \geq \cdots$.

Proof. Let $R = k[x_0, x_1, x_2]$ and $I = I(\mathbb{X})$. Since $\mathbb{X}$ is a finite set of distinct points there is a linear form which misses $\mathbb{X}$ entirely. After a linear change of variables, we may assume this linear form is x_0. Then x_0 is a non-zero-divisor in $A = R/I$. Note that $R/(I, x_0) \cong S/J$ where J the homogeneous ideal obtained by setting $x_0 = 0$ in the generators of I. In addition, $\sqrt{J} = (x_1, x_2)$ and $H(S/J) = \Delta H(\mathbb{X})$. Part (a) can be applied to see that $\Delta H(\mathbb{X}, t) = t + 1$ for $t < \alpha$ and $\Delta H(\mathbb{X}, t) \leq \alpha$ for $t \geq \alpha$.

To see that $\Delta H(\mathbb{X}, \alpha + t) \geq \Delta H(\mathbb{X}, \alpha + t + 1)$ for all $t \geq 0$, note that $S_1 J_{\alpha+t} \subseteq J_{\alpha+t+1}$. Thus, by part (b), $\dim_k(J_{\alpha+t+1}) \geq \dim_k(S_1 J_{\alpha+t}) \geq \dim_k(J_{\alpha+t}) + 1$. Therefore,

$$\begin{aligned}
&\dim_k(J_{\alpha+t+1}) \geq \dim_k(J_{\alpha+t}) + 1 \\
\implies\ & \alpha + t + 1 - \dim_k(J_{\alpha+t}) \geq \alpha + t + 2 - \dim_k(J_{\alpha+t+1}) \\
\implies\ & \dim_k(S_{\alpha+t}) - \dim_k(J_{\alpha+t}) \geq \dim_k(S_{\alpha+t+1}) - \dim_k(J_{\alpha+t+1}) \\
\implies\ & H(S/J, \alpha + t) \geq H(S/J, \alpha + t + 1) \\
\implies\ & \Delta H(\mathbb{X}, \alpha + t) \geq \Delta H(\mathbb{X}, \alpha + t + 1).
\end{aligned}$$

□

(4) Find all possible Hilbert functions for 9 distinct points in $\mathbb{P}^2$. Pick one of the Hilbert functions $\mathcal{H}$ and find a set $\mathbb{X} \subset \mathbb{P}^2$ of 9 distinct points in $\mathbb{P}^2$ such that $H(\mathbb{X}) = \mathcal{H}$. How do you know that the constructed set of points has the selected Hilbert function?

Solution: There are 8 possible Hilbert functions for 9 distinct points in $\mathbb{P}^2$. The possible sequences are listed in the following table:

	$H(\mathbb{X})$	$\Delta H(\mathbb{X})$
A	$(1, 2, 3, 4, 5, 6, 7, 8, 9, 9, \ldots)$	$(1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, \ldots)$
B	$(1, 3, 4, 5, 6, 7, 8, 9, 9, \ldots)$	$(1, 2, 1, 1, 1, 1, 1, 1, 0, 0, \ldots)$
C	$(1, 3, 5, 6, 7, 8, 9, 9, \ldots)$	$(1, 2, 2, 1, 1, 1, 1, 0, 0, \ldots)$
D	$(1, 3, 5, 7, 8, 9, 9, \ldots)$	$(1, 2, 2, 2, 1, 1, 0, 0, \ldots)$
E	$(1, 3, 5, 7, 9, 9, \ldots)$	$(1, 2, 2, 2, 2, 0, 0, \ldots)$
F	$(1, 3, 6, 7, 8, 9, 9, \ldots)$	$(1, 2, 3, 1, 1, 1, 0, 0, \ldots)$
G	$(1, 3, 6, 8, 9, 9, \ldots)$	$(1, 2, 3, 2, 1, 0, 0, \ldots)$
H	$(1, 3, 6, 9, 9, \ldots)$	$(1, 2, 3, 3, 0, 0, \ldots)$

To argue that these are all the possible Hilbert functions, the following facts will be useful:

- $H(\mathbb{X})$ is a differentiable O-sequence with $H(\mathbb{X}, 1) \leq 3$
- $H(\mathbb{X}, d) \leq 9$ for all $d \geq 0$
- $H(\mathbb{X}, d) = 9$ for $d \geq 8$
- $\sum_{t=0}^{8} \Delta H(\mathbb{X}, t) = 9$ (this follows immediately from the above facts)

We know that $1 \leq H(\mathbb{X},1) \leq 3$ and so $1 \leq \Delta H(\mathbb{X},1) \leq 2$. We consider these 2 cases. The details of each case is argued by carefully considering the above facts.

(1) Suppose $\Delta H(\mathbb{X},1) = 1$. Since $\Delta H(\mathbb{X})$ is an O-sequence, $\Delta H(\mathbb{X},t) \leq 1$ for $t \geq 2$. The only possible sequence for $H(\mathbb{X})$ is the sequence in Case A.

(2) Suppose $\Delta H(\mathbb{X},1) = 2$. Then, since $\Delta H(\mathbb{X})$ is an O-sequence, $1 \leq \Delta H(\mathbb{X},2) \leq 3$. This leads to 3 possible situations.
   (i) If $\Delta H(\mathbb{X},2) = 1$ then $\Delta H(\mathbb{X},t) \leq 1$ for $t \geq 2$. This gives Case B.
   (ii) If $\Delta H(\mathbb{X},2) = 2$ then $1 \leq \Delta H(\mathbb{X},3) \leq 2$. If $\Delta H(\mathbb{X},3) = 1$ then we must be in Case C. If $\Delta H(\mathbb{X},3) = 2$ then $1 \leq \Delta H(\mathbb{X},4) \leq 2$: if $\Delta H(\mathbb{X},4) = 1$ then we must be in Case D; if $\Delta H(\mathbb{X},4) = 2$ then we are in Case E.
   (iii) If $\Delta H(\mathbb{X},2) = 3$ then $1 \leq \Delta H(\mathbb{X},3) \leq 4$. If $\Delta H(\mathbb{X},3) = 1$ then we must be in Case F. If $\Delta H(\mathbb{X},3) = 2$ then $\Delta H(\mathbb{X},4) = 1$ giving Case G. If $\Delta H(\mathbb{X},3) = 3$ then $\Delta H(\mathbb{X},4) = 0$ giving Case H. We can never have $\Delta H(\mathbb{X},3) = 4$ since $\sum_{t=0}^{8} \Delta H(\mathbb{X},t) = 9$.

We now concentrate on $\mathcal{H} = (1,3,5,7,8,9,9,\ldots)$. Using the method of lifting monomial ideals with $t_{j,i} = i$ gives $H(\mathbb{X}) = \mathcal{H}$ where

$$\mathbb{X} = \{[1:0:0],[1:1:0],[1:0:1],[1:2:0],[1:1:1],[1:3:0],[1:2:1],[1:4:0],[1:5:0]\}.$$

(5) Suppose that I is a homogeneous ideal in the ring $R = k[x_0,\ldots,x_n]$ where k is an algebraically closed field of characteristic 0. Suppose that $I_d \neq 0$ and that $H(R/I)$ has maximal growth in degree d. Prove that I_d and I_{d+1} have a greatest common divisor of positive degree in the following two cases:
   (a) $n = 1$ and $H(R/I,d) \geq 1$;

   Proof. From our discussion on maximal growth of Hilbert functions it suffices to demonstrate that $PGCD(I_d)$ is positive. We have

$$PGCD(I_d) = \max\{j \mid f_{1,j}(d) \leq H(R/I,d)\},$$

   where by definition

$$f_{1,j} = \binom{d+1}{1} - \binom{d-j+1}{1} = (d+1) - (d-j+1) = j.$$

   Setting $j = 1$, we see that $1 = f_{1,1} \leq H(R/I,d)$. Hence $PGCD(I_d) > 0$ and we are done. □

   (b) $n = 2$ and $H(R/I,d) \geq d+1$.

   Proof. As in part (a), it suffices to demonstrate that $PGCD(I_d)$ is positive. We have

$$PGCD(I_d) = \max\{j \mid f_{2,j}(d) \leq H(R/I,d)\},$$

   where

$$f_{2,j} = \binom{d+2}{2} - \binom{d-j+2}{2}.$$

   Setting $j = 1$, we see that

$$f_{2,1} = \binom{d+2}{2} - \binom{d+1}{2} = d+1 \leq H(R/I,d).$$

   We conclude that $PGCD(I_d) > 0$ which completes the proof. □