Math 314/814 - 006 Exam 2

Blue Exam Solutions

December 4, 2008

Instructor: Dr. S. Cooper

Name: ______________________________

Read each question carefully.

Be sure to show all of your work and not just your final conclusion.

You may not use your notes or text for this exam, but you may use a calculator.

Good Luck!

Problem	Points	Score
1	12	
2	14	
3	8	
4	8	
5	9	
6	12	
7	12	
8	10	
9	15	
$\sum$	100	

(1) Let

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}.$$

For what follows, you can use a calculator to check your work, but you must show work justifying your answer to receive credit.

(a) Without row-reducing, use **Laplace expansion** to find the determinant of A. [8 pts]

Solution: We begin with Laplace expansion down the third column of A:

$$\begin{aligned} \det(A) &= 3 \det \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{bmatrix} \\ &= 3 \left((-1) \det \begin{bmatrix} 2 & 3 \\ 4 & -2 \end{bmatrix} + (3) \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \right) \\ &= 3[(-1)(-4-12) + 3(3-4)] \\ &= 3(16-3) \\ &= 39 \end{aligned}$$

(b) $\det(A^{-1}) = \underline{\frac{1}{\det(A)} = \frac{1}{39}}$ [1 pt]

(c) $\det(A^T) = \underline{\det(A) = 39}$ [1 pt]

(d) $\det(A^2) = \underline{(\det(A))^2 = 39^2 = 1521}$ [1 pt]

(e) $\det(2A) = \underline{2^4 \det(A) = (16)(39) = 624}$ [1 pt]

(2) Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$. For what follows, you can use a calculator to check your work, but you must show work justifying your answer to receive credit.

(a) Find the *eigenvalues* of A. [3 pts]

Solution: We have

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} (2-\lambda) & 1 & 2 \\ 1 & (2-\lambda) & -2 \\ 0 & 0 & (3-\lambda) \end{bmatrix} = (3-\lambda) \det \begin{bmatrix} (2-\lambda) & 1 \\ 1 & (2-\lambda) \end{bmatrix} \\ &= (3-\lambda)(\lambda^2 - 4\lambda + 3) = (3-\lambda)(\lambda-3)(\lambda-1) \end{aligned}$$

We see that the eigenvalues of A are: $\lambda_1 = 3$ (algebraic multiplicity 2) and $\lambda_2 = 1$ (algebraic multiplicity 1).

(b) For each eigenvalue of A, find the corresponding *eigenspace*. [6 pts]

Solution: To find E_3 we solve for $null(A-3I)$. We row reduce the associated augmented matrix:

$$[A - 3I|\mathbf{0}] = \left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Solving the system, we see that $E_3 = Span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right)$.

To find E_1 we solve for $null(A - 1I)$. We row-reduce the associated augmented matrix:

$$[A - 1I|\mathbf{0}] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Solving the system, we see that $E_1 = Span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$.

(c) Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$. If not, explain why it is not. [5 pts]

Solution: We see that the algebraic multiplicity equals the geometric multiplicity for each eigenvalue of A. Thus, A is diagonalizable. We let

$$P = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(3) Let W be a subspace of $\mathbb{R}^n$.

(a) Complete the definition: The *orthogonal complement* of W is the set [2 pts]

Solution: $W^\perp := \{\mathbf{v} \text{ in } \mathbb{R}^n : \mathbf{v}\cdot\mathbf{w} = 0 \text{ for all } \mathbf{w} \text{ in } W\}$.

(b) Find a basis for $W^\perp$ if $W = span\left(\begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right)$. [6 pts]

Solution: $W^\perp = null(A^T)$ where A is the matrix whose columns are the vectors in the spanning set for W. We row-reduce the augmented matrix:

$$[A^T|\mathbf{0}] = \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 0 \\ 2 & -2 & -1 & 3 & 0 \\ -1 & 1 & -1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solving the system, we let $x_4 = t$ and $x_2 = s$. Then $x_3 = x_4 = t$ and $x_1 = x_2 + x_3 - 2x_4 = s - t$.

So, a basis for $W^\perp = null(A^T)$ is $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$.

(4) Let $A = \begin{bmatrix} 2 & 18 & 18 & 38 \\ 2 & 0 & -1 & 1 \\ 1 & 0 & 2 & 3 \end{bmatrix}$. *Fact:* A row reduces to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

Use the Gram-Schmidt Process to find an *orthogonal* basis for the column space of A. [8 pts]

Solution: The first three columns, call these vectors $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ respectively, of A form a basis for $col(A)$. We apply the Gram-Schmidt Process to these to obtain an orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ for $col(A)$.

We start by letting $\mathbf{v_1} = \mathbf{x_1} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. Now let

$$\mathbf{v_2} = \mathbf{x_2} - \frac{\mathbf{x_2}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix} - \frac{36}{9}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -8 \\ -4 \end{bmatrix}.$$

Finally, let

$$\mathbf{v_3} = \mathbf{x_3} - \frac{\mathbf{x_3}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{x_3}\cdot\mathbf{v_2}}{\mathbf{v_2}\cdot\mathbf{v_2}}\mathbf{v_2} = \begin{bmatrix} 18 \\ -1 \\ 2 \end{bmatrix} - \frac{36}{9}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \frac{180}{180}\begin{bmatrix} 10 \\ -8 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

(5) Consider the inconsistent system

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 2 \\ x - 2y &= 0 \end{aligned}$$

(a) What are the normal equations for the least squares approximating solution? [3 pts]

Solution: Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Then

$$A^T A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

and

$$A^T \mathbf{b} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

So, the normal equations are

$$A^T A \overline{\mathbf{x}} = A^T \mathbf{b} \text{ or } \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

(b) Find the least squares approximating solution. [6 pts]

Solution: We need to solve the system given by the normal equations in part (a). We row-reduce the associated augmented matrix:

$$\begin{bmatrix} 6 & -3 & | & 4 \\ -3 & 6 & | & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 6 & -3 & | & 4 \\ 0 & 9/2 & | & 3 \end{bmatrix}.$$

Solving the system, we see that

$$\overline{\mathbf{x}} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}.$$

(6) For each of the following, determine whether W is a subspace of M_{22} (using the usual definitions of matrix addition and scalar multiplication). If W is a subspace of M_{22}, then find $\dim(W)$. If W is not a subspace of M_{22}, give an explicit example showing how it fails to be one. [6 pts each]

(a) $W = \{A \in M_{22} : \det(A) = 0\}$

Solution: W is not a subspace of M_{22}. To see this, note that $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both in W, but $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not in W since $\det(A + B) = 1 \neq 0$.

(b) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b = c + d \right\}$

Solution: W is a subspace of M_{22}. To see this, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ be in W and α be a scalar. Then

$$A + B = \begin{bmatrix} (a+e) & (b+f) \\ (c+g) & (d+h) \end{bmatrix}$$

and $(a + e) + (b + f) = (a + b) + (e + f) = (c + d) + (g + h) = (c + g) + (d + h)$. We conclude that $A + B$ is in W. Also,

$$\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$$

and $\alpha a + \alpha b = \alpha(a + b) = \alpha(c + d) = \alpha c + \alpha d$. This shows that αA is also in W.

To find $\dim(W)$, we need to find a basis for W. Notice that

$$\begin{aligned} W &= \left\{ \begin{bmatrix} c+d-b & b \\ c & d \end{bmatrix} \right\} \\ &= \left\{ c \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \mathit{Span} \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \right) \end{aligned}$$

By inspection, if c_1, c_2 and c_3 are scalars such that

$$c_1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then $c_1 = c_2 = c_3 = 0$. Thus, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is a basis for W.

Therefore, $\dim(W) = 3$.

(7) The sets $\mathcal{B} = \{1, 1+x, (1+x)^2\}$ and $\mathcal{C} = \{1+x, x+x^2, 1+x^2\}$ are two bases for $\mathcal{P}_2$.

(a) Find the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$. Be sure to show all of your work. [9 pts]

Solution: By inspection, we see that

$$\begin{aligned} 1 &= \frac{1}{2}(1+x) - \frac{1}{2}(x+x^2) + \frac{1}{2}(1+x^2) \\ 1+x &= 1(1+x) + 0(x+x^2) + 0(1+x^2) \\ 1+2x+x^2 &= 1(1+x) + 1(1+x^2) + 0(1+x^2) \end{aligned}$$

So, by definition,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1/2 & 1 & 1 \\ -1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix}.$$

(b) Use your answer to part (a) to express $p(x) = 7 + 5x + 4x^2$ as a linear combination of the polynomials in $\mathcal{C}$. [3 pts]

Solution: By inspection, we see that

$$p(x) = 7 + 5x + 4x^2 = 4(1+2x+x^2) - 3(1+x) + 6(1).$$

So,

$$[p(x)]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[p(x)]_{\mathcal{B}} = \begin{bmatrix} 1/2 & 1 & 1 \\ -1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}.$$

Thus,

$$p(x) = 4(1+x) + 1(1+x^2) + 3(1+x^2).$$

(8) Let V and W be vector spaces. Suppose that $T : V \to W$ is a transformation.

(a) Complete the definition: $T : V \to W$ is a *linear transformation* if [2 pts]

Solution: $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(c\mathbf{x}) = cT(\mathbf{x})$ for all $\mathbf{x}, \mathbf{y}$ in V and all scalars c.

(b) Suppose that $T : V \to W$ is a linear transformation and let $V = Span(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k})$. Prove that the range of T equals $Span(T(\mathbf{v_1}), T(\mathbf{v_2}), \ldots, T(\mathbf{v_k}))$. [5 pts]

Solution:

$$\begin{aligned} Range(T) &= \{T(\mathbf{x}) : \mathbf{x} \in V\} \\ &= \{T(c_1\mathbf{v_1} + \cdots c_k\mathbf{v_k}) : \mathbf{x} = c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k}\} \\ &= \{c_1T(\mathbf{v_1}) + \cdots + c_kT(\mathbf{v_k})\} \\ &= Span(T(\mathbf{v_1}), T(\mathbf{v_2}), \ldots, T(\mathbf{v_k})) \end{aligned}$$

(c) Let $T : \mathcal{P}_2 \to M_{22}$ be the linear transformation such that

$$T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T(1+x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T(1+x+x^2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

You may assume that $\{1, 1+x, 1+x+x^2\}$ is a basis for $\mathcal{P}_2$. Find $T(5-3x+2x^2)$. [3 pts]

Solution: Note that

$$5 - 3x + 2x^2 = 2(1+x+x^2) - 5(1+x) + 8(1).$$

So,

$$\begin{aligned} T(5-3x+2x^2) &= 2T(1+x+x^2) - 5T(1+x) + 8T(1) \\ &= 2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - 5\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -7 \\ 2 & 3 \end{bmatrix} \end{aligned}$$

(9) Are the following statements *true* or *false*? Carefully justify your answers. [3 pts each]

(a) If the $n \times n$ invertible matrix A has eigenvalue $\lambda = 2$, then A^{-1} has eigenvalue 2.

Solution: This statement is false. For example, let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$. We see that 2 is an eigenvalue of A but not of A^{-1}.

(b) If A is an orthogonal matrix, then $\det(A^{-1}) = \det(A)$.

Solution: This statement is true. If A is orthogonal, then $A^{-1} = A^T$. So,

$$\det(A^{-1}) = \det(A^T) = \det(A).$$

(c) If the 4×4 matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ each having algebraic and geometric multiplicity 2, then $A^{50} = I_4$.

Solution: This statement is true. The matrix A must be diagonalizable. Let P be the matrix of linearly independent eigenvectors for A and D be the corresponding diagonal matrix whose diagonal entries are the eigenvalues of A. Then $D^{50} = I_4$ and

$$A = PDP^{-1} \implies A^{50} = PD^{50}P^{-1} = PI_4P^{-1} = I_4.$$

(d) Suppose $T : V \to W$ is a linear transformation. If $\mathbf{v_1}, \ldots, \mathbf{v_k}$ are linearly dependent vectors in V, then $T(\mathbf{v_1}), \ldots, T(\mathbf{v_k})$ are also linearly dependent.

Solution: This statement is true. Since $\mathbf{v_1}, \ldots, \mathbf{v_k}$ are linearly dependent there exist scalars $c_1, \ldots, c_k$ not all zero such that $c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$. Thus

$$T(c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k}) = T(\mathbf{0}) \implies c_1T(\mathbf{v_1}) + \cdots + c_kT(\mathbf{v_k}) = \mathbf{0}.$$

Since at least one of the scalars is non-zero, we see that $T(\mathbf{v_1}), \ldots, T(\mathbf{v_k})$ are also linearly dependent.

(e) $\mathcal{B} = \{1 + x, 2 - x + x^2, 3x - 2x^2, -1 + 3x + x^2\}$ is a basis for $\mathcal{P}_2$.

Solution: This statement is false. $\mathcal{B}$ has 4 vectors, but $\dim(\mathcal{P}_2) = 3$.