

## Math 314/814, Fall 2008

### Solutions to Application Mini-Project #3: Perron, Frobenius and Big 12 Soccer

We begin by importing the LinAlg package and setting the display matrix size to 12x12.

```
> with(linalg);
```

```
[BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, (1)
  addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat,
  charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto,
  crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals,
  eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim,
  fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad,
  hadamard, hermite, hessian, hilbert, htranspose, ihermite, indexfunc, innerprod, intbasis,
  inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve,
  matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace,
  orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim,
  rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix, subvector,
  sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent,
  vectdim, vector, wronskian]
```

```
> interface(rtablesiz = 12);
```

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(2)

(1). We let  $A$  be the matrix below which records the results of the games. Starting with 1, the rows and columns of  $A$  correspond to the teams: Baylor, Colorado, Iowa State, Kansas, Missouri, Nebraska, Oklahoma, Oklahoma State, Texas, Texas A&M, and Texas Tech, respectively. The  $(i,j)$  entry of  $A$  is 1 if team  $i$  defeats team  $j$  and 0 otherwise. So, for example, Baylor defeated Nebraska only resulting in a 1 in the  $(1,6)$  entry of  $A$  and 0 for the remaining first row entries of  $A$ . Also, Oklahoma State defeated Baylor, Missouri, Nebraska, Oklahoma and Texas. This results in a 1 in the  $(8, 1)$ ,  $(8, 5)$ ,  $(8, 6)$ ,  $(8,7)$ ,  $(8,9)$  entries of  $A$  and 0 for the remaining row 8 entries of  $A$ .

```
> A := Matrix([ [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0], [1, 0, 0, 0, 1, 0, 1, 1,
  0, 0, 0], [0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1], [1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1], [0, 0, 0, 0, 0, 0, 1, 0,
  0, 0, 0], [1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0], [1, 1, 0, 0, 1, 1, 1, 0,
  0, 0, 1], [1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1], [1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0]]);
```

(3)

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (3)$$

To show that  $A$  satisfies the hypotheses of the Perron-Frobenius Theorem, we need to verify that  $A$  is nonnegative and irreducible. Since every entry of  $A$  is nonnegative, we see that indeed  $A$  is nonnegative. To show that  $A$  is irreducible we verify that it is primitive (it is given that every primitive matrix is irreducible on page 332 of the text). That is, we need to verify that  $A^k > 0$  for some power  $k$  (i.e., every entry of  $A^k$  is positive for some power  $k$ ). We consider the 8th power of the matrix  $A$  below.

**> A^8;**

$$\begin{bmatrix} 123 & 61 & 52 & 40 & 42 & 126 & 114 & 51 & 15 & 20 & 45 \\ 2310 & 1015 & 959 & 772 & 926 & 2354 & 2136 & 980 & 381 & 274 & 755 \\ 2670 & 1127 & 1037 & 870 & 1104 & 2676 & 2476 & 1097 & 449 & 327 & 848 \\ 3847 & 1680 & 1442 & 1292 & 1529 & 3885 & 3539 & 1489 & 630 & 529 & 1291 \\ 4109 & 1779 & 1559 & 1397 & 1623 & 4152 & 3771 & 1611 & 690 & 547 & 1362 \\ 400 & 162 & 166 & 134 & 158 & 397 & 367 & 178 & 71 & 42 & 117 \\ 1261 & 522 & 455 & 409 & 549 & 1262 & 1174 & 487 & 220 & 158 & 405 \\ 2972 & 1236 & 1106 & 957 & 1239 & 2952 & 2758 & 1190 & 499 & 379 & 938 \\ 3452 & 1488 & 1278 & 1157 & 1380 & 3473 & 3175 & 1332 & 573 & 472 & 1145 \\ 6208 & 2671 & 2382 & 2068 & 2540 & 6271 & 5736 & 2475 & 1044 & 790 & 2037 \\ 2640 & 1188 & 1071 & 895 & 1042 & 2708 & 2434 & 1076 & 427 & 339 & 898 \end{bmatrix} \quad (4)$$

Since every entry of  $A^8$  is positive, we conclude that  $A$  is primitive, and hence irreducible. Thus, by the Perron Frobenius Theorem,  $A$  has a unique ranking vector  $r$ . This vector  $r$  is an eigenvector of  $A$  which is associated to a real eigenvalue. To find the eigenvalues of  $A$  we use the command `evalf(eigenvalues(A))`:

**> evalf(eigenvalues(A));**

$$\begin{aligned} & 3.165397774, 0.4345879580 + 0.6763296374 I, -0.2633085350 + 1.776019965 I, \\ & -0.7202960127 + 1.368065823 I, -0.5122407669 + 0.9030958218 I, -0.5214415302 \\ & + 0.1494422190 I, -0.5214415302 - 0.1494422190 I, -0.5122407669 - 0.9030958218 I, \\ & -0.7202960127 - 1.368065823 I, -0.2633085350 - 1.776019965 I, 0.4345879580 \end{aligned} \quad (5)$$

– 0.6763296374 I

The only real eigenvalue of A is the first one listed: 3.165397774. By the Perron-Frobenius Theorem the algebraic and hence geometric multiplicity equals 1. To find a basis vector for the associated eigenspace we use the command `evalf(eigenvectors(A)[1])`:

```
> evalf(eigenvectors(A)[1]);
[3.165397774, 1., {[1., 18.63397, 21.2749664, 30.71647240, 32.800229, 3.165397774,
10.01974307, 23.523752, 27.47668565, 49.651356, 21.35528]}] (6)
```

The output first gives the eigenvalue, then its algebraic multiplicity and then a basis for the associated eigenspace. We can take this basis vector as the ranking vector! Thus, we would rank the teams as follows (from strongest to weakest ranking): Texas A&M, Missouri, Kansas, Texas, Oklahoma State, Texas Tech, Iowa State, Colorado, Oklahoma, Nebraska, Baylor. The highest ranked team corresponds to the largest component of the eigenvector, the second highest ranked team corresponds to the second highest largest component of the eigenvector, and so on.

(2). We now repeat the above process, but we put 0.5 in the entries of A for the games that ended in ties while leaving all the other entries of A unchanged. This gives us the following matrix:

```
> B := Matrix([ [0, 0, 0, 0.5, 0, 1, 0, 0, 0, 0, 0], [1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0], [1, 0, 0, 0, 1, 0.5,
1, 1, 0.5, 0, 0], [0.5, 1, 1, 0, 1, 1, 0, 0.5, 0.5, 0, 1], [1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1], [0, 0, 0.5,
0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0.5], [1, 0, 0, 0.5, 1, 1, 1, 0, 1, 0, 0], [1, 1,
0.5, 0.5, 1, 1, 1, 0, 0, 0, 1], [1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1], [1, 1, 1, 0, 0, 1, 0.5, 1, 0, 0, 0]]);
```

$$B := \begin{bmatrix} 0 & 0 & 0 & 0.5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0.5 & 1 & 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 1 & 0 & 1 & 1 & 0 & 0.5 & 0.5 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 1 & 0 & 0 & 0.5 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0.5 & 0.5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0.5 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (7)$$

Again, to show that B satisfies the hypotheses of the Perron-Frobenius Theorem, we need to verify that B is nonnegative and irreducible. Since every entry of B is nonnegative, we see that indeed B is nonnegative. To show that B is irreducible we verify that it is primitive. We consider the 8th power of the matrix B below.

```
> B^8;
[[2953.58984375000000, 1354.44531250000000, 1625.75000000000000,
1433.14843750000000, 1339.91406250000000, 3002.96875000000000,
2677.10937500000000, 1361.60546875000000, 823.81640625000000,
343.70312500000000, 1356.50781250000000],
[9584.57812500000000, 4351.09375000000000, 5330.58593750000000,
```

4588.70312500000000, 4316.21875000000000, 9728.89843750000000,  
 8716.82812500000000, 4531.98437500000000, 2683.35156250000000,  
 1080.21875000000000, 4304.41406250000000 ],  
 [ 11881.42968750000000, 5335.25781250000000, 6491.95312500000000,  
 5723.80859375000000, 5419.53906250000000, 12031.14062500000000,  
 10813.98437500000000, 5516.96875000000000, 3350.14843750000000,  
 1350.07812500000000, 5337.92187500000000 ],  
 [ 16849.66406250000000, 7734.71093750000000, 9300.94140625000000,  
 8137.63671875000000, 7628.43750000000000, 17133.51562500000000,  
 15280.08593750000000, 7808.67968750000000, 4690.55468750000000,  
 1960.98437500000000, 7717.85156250000000 ],  
 [ 14416.73437500000000, 6630.53125000000000, 7927.47656250000000,  
 6989.07031250000000, 6493.78125000000000, 14647.03125000000000,  
 13066.52343750000000, 6634.37500000000000, 3993.40625000000000,  
 1704.40625000000000, 6637.27343750000000 ],  
 [ 3174.59375000000000, 1429.73437500000000, 1750.72656250000000,  
 1521.50000000000000, 1430.10156250000000, 3212.84765625000000,  
 2890.41406250000000, 1492.64062500000000, 889.48046875000000,  
 359.42187500000000, 1420.21093750000000 ],  
 [ 6482.76562500000000, 2953.08593750000000, 3536.05468750000000,  
 3166.28906250000000, 2944.57812500000000, 6577.47656250000000,  
 5868.06640625000000, 2962.35937500000000, 1813.10156250000000,  
 755.06250000000000, 2976.75000000000000 ],  
 [ 13116.66796875000000, 5900.38281250000000, 7126.07031250000000,  
 6352.25781250000000, 6000.49218750000000, 13281.42187500000000,  
 11924.98437500000000, 6027.08984375000000, 3695.30078125000000,  
 1507.95312500000000, 5935.51562500000000 ],  
 [ 15841.40234375000000, 7256.36718750000000, 8693.99218750000000,  
 7673.64062500000000, 7178.12500000000000, 16091.61328125000000,  
 14364.31250000000000, 7290.73046875000000, 4409.12500000000000,  
 1857.40625000000000, 7268.82812500000000 ],  
 [ 23208.88281250000000, 10596.32812500000000, 12798.81250000000000,  
 11220.63281250000000, 10494.60937500000000, 23570.05468750000000,  
 21051.42968750000000, 10778.00781250000000, 6478.85937500000000,  
 2675.59375000000000, 10578.63281250000000 ],  
 [ 11177.89843750000000, 5133.25000000000000, 6239.57031250000000,  
 5362.91406250000000, 5018.90625000000000, 11371.07031250000000,  
 10145.79687500000000, 5267.62500000000000, 3108.44531250000000,  
 1279.78125000000000, 5077.41015625000000 ] ]

Since every entry of  $B^8$  is positive, we conclude that  $B$  is primitive, and hence irreducible. Thus, by

the Perron Frobenius Theorem, B has a unique ranking vector r. This vector r is an eigenvector of B which is associated to a real eigenvalue. To find the eigenvalues of B we use the command evalf (eigenvalues(B)):

```
> evalf(eigenvalues(B));
3.921325048, -0.4672863111 + 1.923999633 I, -0.4672863111 - 1.923999633 I,
-0.4665322519 + 1.472164304 I, -0.4665322519 - 1.472164304 I, -0.4850938389
+ 0.9588321636 I, -0.4850938389 - 0.9588321636 I, -0.09091726043
+ 0.5329720996 I, -0.09091726043 - 0.5329720996 I, -0.4158392802, -0.4858264429
```

(9)

In this case, the matrix B has 3 real eigenvalues. By the Perron-Frobenius Theorem, we want to use the eigenvalue which is largest in absolute value. So, we select the eigenvalue 3.921325048. By the Perron-Frobenius Theorem the algebraic and hence geometric multiplicity equals 1. To find a basis vector for the associated eigenspace we use the command evalf(eigenvectors(B)[10]):

```
> L:=evalf(eigenvectors(B)[10]);
L:= [3.921325039, 1., {[ -0.1829293909, -0.5920694578, -0.7327924798, -1.043289326,
-0.8923244398, -0.1956809383, -0.4009323251, -0.8093616693, -0.980258436,
-1.435547540, -0.6919345028 ]}]
```

(10)

We would like to work with a positive eigenvector, so we multiply the components by -1. This gives the ranking vector

```
> M:=Matrix([[0.1829293909],[0.5920694578],[0.7327924798],
[1.043289326],[0.8923244398],[0.1956809383],[0.4009323251],
[0.8093616693],[0.980258436],[1.435547540],[0.6919345028]]);
```

$$M := \begin{bmatrix} 0.1829293909 \\ 0.5920694578 \\ 0.7327924798 \\ 1.043289326 \\ 0.8923244398 \\ 0.1956809383 \\ 0.4009323251 \\ 0.8093616693 \\ 0.980258436 \\ 1.435547540 \\ 0.6919345028 \end{bmatrix} \quad (11)$$

Thus, we would rank the teams as follows (from strongest to weakest ranking): Texas A&M, Kansas, Texas, Missouri, Oklahoma State, Iowa State, Texas Tech, Colorado, Oklahoma, Nebraska, Baylor. The highest ranked team corresponds to the largest component of the eigenvector, the second highest ranked team corresponds to the second highest largest component of the eigenvector, and so on.

(3). Let  $r_i$  denote the ranking of team  $i$ , where we order the teams as described in the matrices A and B above. As in the example on pages 353 - 354, we insist that the  $i$ th team's ranking should be proportional to the sum of the rankings of the teams defeated and tied by the team  $i$ .

Letting  $c$  be the constant of proportionality, we obtain a system of linear equations.

For example, the ranking of Baylor (team 1) has ranking  $r_1$ . Baylor defeated Nebraska (team 6) and tied Kansas (team 4). This leads to the equation

$$r_1 = c(0.5r_4 + r_6).$$

Repeating the idea for each of the 11 teams, we obtain the linear system:

$$\begin{aligned} r_1 &= c(0.5r_4 + r_6) \\ r_2 &= c(r_1 + r_3 + r_6 + r_7 + r_8) \\ r_3 &= c(r_1 + r_5 + 0.5r_6 + r_7 + r_8 + 0.5r_9) \\ r_4 &= c(0.5r_1 + r_2 + r_3 + r_5 + r_6 + 0.5r_8 + 0.5r_9) \\ r_5 &= c(r_1 + r_2 + r_6 + r_7 + r_{10} + r_{11}) \\ r_6 &= c(0.5r_3 + r_7) \\ r_7 &= c(r_1 + r_4 + 0.5r_{11}) \\ r_8 &= c(r_1 + 0.5r_4 + r_5 + r_6 + r_7 + r_9) \\ r_9 &= c(r_1 + r_2 + 0.5r_3 + 0.5r_4 + r_5 + r_6 + r_7 + r_{11}) \\ r_{10} &= c(r_1 + r_2 + r_3 + r_4 + r_6 + r_7 + r_8 + r_9 + r_{11}) \\ r_{11} &= c(r_1 + r_2 + r_3 + r_6 + 0.5r_7 + r_8) \end{aligned}$$

The matrix  $B$  above is the coefficient matrix of this system. Let  $\mathbf{r}$  be the column vector given by the  $r_i$ . Then the above system is equivalent to

$$\mathbf{r} = c B \mathbf{r}.$$

In other words, the vector  $\mathbf{r}$  is an eigenvector of the matrix  $B$  with eigenvalue  $1/c$ . As noted in Task (2), we can apply the Perron-Frobenius Theorem to guarantee a unique ranking vector  $\mathbf{r}$ .

This results in a reasonable ranking because teams which tie are now treated equally under comparison and hence their rankings depend on each other. For example, by substituting  $r_4$  into  $r_1$  we obtain the relationship

$$r_1 = c(0.25r_1 + 0.5r_2 + 0.5r_3 + 1.5r_6 + 0.5r_7 + 0.5r_{10} + 0.5r_{11})$$

Ignoring ties,  $r_1$  depends only on  $r_6$  (the ranking of Nebraska). However, including information on ties indicates that the ranking of Baylor depends on the ranking of many more teams. In fact, one gets the impression that Baylor is a stronger team than what is inferred in the first task since it ties Kansas which defeats many other stronger teams than Nebraska.

(4). In my opinion, the ranking in (2) seems to be more reasonable. For example, in (1) Missouri was ranked stronger than Kansas whereas in (2) Kansas was ranked stronger than Missouri. One can argue that indeed Kansas is the stronger team since it tied both Oklahoma State and Texas which both defeated Missouri. Not taking ties into account would ignore this data. Answers to this task will vary depending on opinions.

(5). Using the same ideas as in parts (1) and (2), we obtain the following matrix where the first 11 rows and columns represent the same teams as above and the 12th row and 12 column represent the team for Kansas State. Note that when two teams do not play each other we place a 0 in the corresponding entry of the matrix.

```
> C:=Matrix([[0,0,1,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0,0,1],
[0,0,0,0,0,0,0,0,0,0,0,0],[0,1,1,0,0,0,0,0,0,0,0,0],[0,1,0,0,0,
1,0,0,0,0,0,0],[1,0,1,0,0,0,0,0,0,0,0,0],[1,0,0,1,0,0,0,0,0,0,
0,1],[1,0,0,0,1,0,0,0,0,1,0,0],[0,1,0,0,1,0,1,1,0,0,0,0],[0,0,
1,0,0,0,0,0,0,0,0,0],[0,0,0,1,0,1,0,0,0,1,0,1],[0,0,0,0,0,0,0,0,
```

`0,0,1,0,0]]);`

$$C := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (12)$$

Note that if we interchange first rows and columns 2 and 11, then rows and columns 3 and 8, and finally rows and columns 8 and 10, we obtain the matrix:

`> L:=Matrix([[0,0,0,0,0,0,0,0,1,0,0,0,0],[0,0,0,0,0,1,0,1,0,0,0,0,1],  
[1,0,0,0,0,1,0,0,0,0,1,0,0],[0,0,0,0,0,0,0,0,1,0,0,1,0],[0,0,0,0,0,  
1,0,0,0,0,1,0],[1,0,0,0,0,0,0,0,1,0,0,0,0],[1,0,0,1,0,0,0,0,0,0,0,  
0,1],[0,0,0,0,0,0,0,0,1,0,0,0,0],[0,0,1,0,1,0,1,0,0,0,1,0],[0,0,  
0,0,0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0,0,0,1],[0,0,0,0,0,0,0,  
0,0,1,0,0]]);`

$$L := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (13)$$

This shows that C is reducible by definition. Therefore, we cannot apply the Perron-Frobenius Theorem. So it is not guaranteed that C has a unique ranking eigenvector. Indeed, since there can be

two linearly independent ranking vectors then once can argue that the resulting rankings are meaningless. In fact, the only eigenvalue of the matrix  $C$  is 0 and its eigenspace is described below:

$$\begin{aligned}
 &> \text{eigenvectors}(C); \\
 &[0, 12, \{ [ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -1 \ 1 \ 0 \ 0 \ 0 \ 0 ], [ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 ], \\
 &\quad [ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 ] \}]
 \end{aligned} \tag{14}$$

We see that the associated eigenspace has dimension 3 which results in two nonnegative linearly independent ranking vectors. It is unclear which ranking vector is meaningful.