Rounding Issues

Math 314—006
Application Mini-Project #1
Solutions

1. Read the four paragraphs on page 66, and do the four problems on pages 66–67. Note that #1 and #2 are computational. This means you need to show your work clearly, and you may decide that a sentence or two of explanation is needed. On the other hand, #3 and #4 require you to give careful and thoughtful explanations.

   Solution:

   p. 66, #1 We wish to solve the following system exactly:

$$\begin{array}{rcrcl} x & + & y & = & 0 \\ x & + & \frac{801}{800}y & = & 1 \end{array}$$

   ¿From the first equation, we have $x = -y$. Plugging this into the second equation gives

$$\begin{aligned} -y + \frac{801}{800}y &= 1 \\ \frac{1}{800}y &= 1 \\ y &= 800 \end{aligned}$$

   and so the solution is $(x, y) = (-800, 800)$.

   p. 66, #2 Now we wish to solve

$$\begin{array}{rcrcl} x & + & y & = & 0 \\ x & + & 1.0012y & = & 1 \end{array}$$

   by rounding the result of each calculation to 5 significant digits. Again, the first equation gives us $x = -y$ and plugging into the second gives

$$\begin{aligned} -y + 1.0012y &= 1 \\ .0012y &= 1 \\ y &= 833.33 \end{aligned}$$

   and so the solution is $(x, y) = (-833.33, 833.33)$.

   p. 67, #3 When we round to four significant digits, we still have $x = -y$ from the first equation and plugging in gives

$$\begin{aligned} -y + 1.001y &= 1 \\ .001y &= 1 \\ y &= 1000 \end{aligned}$$

and so the solution is $(x, y) = (-1000, 1000)$.

When we round to three significant digits, we have:

$$\begin{aligned} -y + 1.00y &= 1 \\ 0 &= 1 \end{aligned}$$

and so when we round to three significant digits, the system has no solution.

The upshot is that the fewer number of significant digits we use, the less accurate our solution is. Moreover, in this particular example, a very small rounding error leads to a huge difference in the solution set.

p. 67, #4 In the exact system, we have the graphs of two lines: $y = -x$ and $y = -\frac{800}{801}x + 1$. One has slope -1 and the other has slope $-\frac{800}{801} \approx -0.99875$. Although the slopes of these two lines are close, the lines are not parallel and therefore they intersect.

When we round to five significant digits, the second line becomes $y = -\frac{1}{1.0012}x + 1$, which has a slope of $-\frac{1}{1.0012} \approx -0.99880$. Thus the slopes are even closer to each other. Moreover, since the slopes of the original two lines were very close, a small rounding error led to the point of intersection to move significantly.

When we round to four significant digits, the slope of the second line becomes $-\frac{1}{1.001} \approx -0.99900$, which is even closer to the slope of the first line. Again, because of the closeness of the slopes, a small rounding error led to a big jump in the point of intersection.

When we round to three significant digits, the slope of the second line becomes -1, which is the same as the slope of the first line. Therefore the lines are now parallel and so they cannot intersect, which is why we have no solutions.

2. Read the paragraph on page 67, and, as suggested, do some experiments with the linear system given there. Be sure to write a nice description of what you see happening as well as an explanation of why you think this is happening.

   Solution: The system we start with is

$$\begin{array}{rcrcl} 4.552x &+& 7.083y &=& 1.931 \\ 1.731x &+& 2.693y &=& 2.001 \end{array}$$

   We solve this by first solving for x in terms of y in the second equation:

$$x = \frac{2.001}{1.731} - \frac{3.693}{1.731}y$$

   We then plug this into the first equation, find a value for y, and then plug that back in to get a value for x.

Rounding to 8 significant digits at each step gives

$$\begin{aligned}
x &= 1.1559792 - 1.5557481y \\
4.552(1.1559792 - 1.5557481y) + 7.083y &= 1.931 \\
y &= \frac{1.931 - 4.552 \cdot 1.1559792}{7.083 - 4.5521.5557481} \\
&= \frac{-3.3310173}{.0012346400} \\
&= -2697.9655 \\
x &= 1.1559792 - 1.5557481 \cdot (-2697.9655) \\
&= 4198.5122
\end{aligned}$$

Rounding to 7 significant digits at each step gives

$$\begin{aligned}
x &= 1.155979 - 1.555748y \\
4.552(1.155979 - 1.555748y) + 7.083y &= 1.931 \\
y &= \frac{1.931 - 4.552 \cdot 1.155979}{7.083 - 4.5521.555748} \\
&= \frac{-3.331016}{.001235100} \\
&= -2696.961 \\
x &= 1.155979 - 1.555748 \cdot (-2696.961) \\
&= 4196.947
\end{aligned}$$

Rounding to 6 significant digits at each step gives

$$\begin{aligned}
x &= 1.15598 - 1.55575y \\
4.552(1.15598 - 1.55575y) + 7.083y &= 1.931 \\
y &= \frac{1.931 - 4.552 \cdot 1.15598}{7.083 - 4.5521.55575} \\
&= \frac{-3.33102}{.00122500} \\
&= -2719.20 \\
x &= 1.15598 - 1.55575 \cdot (-2719.20) \\
&= 4231.55
\end{aligned}$$

Rounding to 5 significant digits at each step gives

$$\begin{aligned}
x &= 1.1560 - 1.5557y \\
4.552(1.1560 - 1.5557y) + 7.083y &= 1.931 \\
y &= \frac{1.931 - 4.552 \cdot 1.1560}{7.083 - 4.5521.5557} \\
&= \frac{-3.3311}{.001450} \\
&= -2297.3 \\
x &= 1.1560 - 1.5557 \cdot (-2297.3) \\
&= 3575.1
\end{aligned}$$

Rounding to 4 significant digits at each step gives

$$\begin{aligned}
x &= 1.156 - 1.556y \\
4.552(1.156 - 1.556y) + 7.083y &= 1.931 \\
y &= \frac{1.931 - 4.552 \cdot 1.156}{7.083 - 4.5521.556} \\
&= \frac{-3.331}{0}
\end{aligned}$$

and so the system is inconsistent.

Again, we can explain this geometrically. The original system represents two lines with slopes $-0.6426655372\ldots$ and $-0.6427775715\ldots$, respectively. These slopes are very close together, and rounding eventually makes them parallel.

3. Read the paragraph at the top of page 86 and do the two problems on that page. Again, #1 is a computational problem and so you need to show your work clearly, and you may decide that a sentence or two of explanation is needed. The questions at the end of each part of #2 require more thoughtful and careful explanations.

Solution:

p. 86, #1 (a) We get $x = \frac{1}{0.00021} = \frac{100000}{21} = 4761.90476190476190\ldots.$

p. 86, #1 (b) (*Note*: The author of our text used a funny notion of "significant digits" here. We'll take him to mean "number of digits after the decimal", at least for this problem.) Rounding gives $x = \frac{1}{.0002} = 5000$.

p. 86, #2 (a) The mathematics of this one were essentially done in the "Clarification" passed out. We have:

$$\left[\begin{array}{rr|r} 0.400 & 99.6 & 100 \\ 75.3 & -45.3 & 30 \end{array}\right] \xrightarrow{R_2' = R_2 - \frac{75.3}{.4}R_1} \left[\begin{array}{rr|r} 0.400 & 99.6 & 100 \\ 0 & -18700 & -18800 \end{array}\right]$$

This is in row reduced form, and so our Gaussian elimination is complete. We now use back-substitution. The second row tells us that $-18700y = -18800$ and so $y = 1.01$ (to three significant digits). Plugging into the top row gives

$$\begin{aligned} 0.400x + (99.6)(1.01) &= 100 \\ x + \frac{(99.6)(1.01)}{.400} &= \frac{100}{.400} \\ x + 251 &= 250 \\ x &= -1 \end{aligned}$$

The error is caused by the need to take a large multiple (namely, 75.3/.4) of the entries in row one and adding them to the comparatively small entries in row two. Since we're rounding off, the entries in row two get washed out of the calculation, resulting in an error.

p. 86, #2 (b) We flip the two rows and row-reduce:

$$\left[\begin{array}{cc|c} 75.3 & -45.3 & 30.0 \\ 0.400 & 99.6 & 100 \end{array}\right] \xrightarrow{R_2' = R_2 - \frac{.400}{75.3}R_1} \left[\begin{array}{cc|c} 75.3 & -45.3 & 30.0 \\ 0 & 99.8 & 99.8 \end{array}\right]$$

This is in row reduced form, and so our Gaussian elimination is complete. We now use back-substitution. The second row tells us that $99.8y = 99.8$ and so $y = 1.00$ (to three significant digits). Plugging into the top row gives

$$\begin{aligned} 75.3x - (45.3)(1) &= 30.0 \\ x + \frac{-45.3}{75.3} &= \frac{30.0}{75.3} \\ x - 0.602 &= 0.398 \\ x &= 1.00 \end{aligned}$$

So flipping the rows gives us the correct answer. The point is that we're no longer having to take a large multiple of the entries in row one and adding them to the comparatively small entries in row two.

4. Read the paragraph at the bottom of page 86 and then do #3, on page 87. Although this problem is stated as a purely computational one, you should also write a sentence or two of explanation at the end, describing what you found.

Solution:

p. 87, #3(a) We mimic the "Clarification" that was handed out, starting with the augmented matrix

$$\left[\begin{array}{cc|c} .001 & .995 & 1.00 \\ -10.2 & 1.00 & -50.0 \end{array}\right]$$

Our first row operation is $R_2' = R_2 + \frac{10.2}{.001}R_1$. The first entry of R_2' is 0. For the second, we first round $\frac{10.2}{.001} * .995 = 10149$ to 3 significant digits and get 10100. Adding this to the second entry of R_2 gives 10101, which we round to 10100. The third entry is $-50 + \frac{10.2}{.001}(1) = 10150$, which we round to 10200. So we have

$$\left[\begin{array}{cc|c} .001 & .995 & 1.00 \\ -10.2 & 1.00 & -50.0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} .001 & .995 & 1.00 \\ 0 & 10100 & 10200 \end{array}\right]$$

This is in row echelon form and so we use back-substitution to get the solution $x = -4.95$, $y = 1.01$ (with all results rounded to three significant digits).

Now we try it with the rows switched:

$$\left[\begin{array}{rr|r} -10.2 & 1.00 & -50.0 \\ .001 & .995 & 1.00 \end{array}\right] \to \left[\begin{array}{rr|r} -10.2 & 1.00 & -50.0 \\ 0 & .995 & .995 \end{array}\right]$$

This is in row-echelon form and back-substitution yields $x = 5.00$, $y = 1.00$.

We see that we get the correct solution when we use partial pivoting but not when we don't. As before, this is because when we don't use partial pivoting, we end up multiplying by such a large number that we change the scale, and then rounding ends up eliminating the information from one of the equations.

p. 87, #3(b) We start with the rows in the order given and round to 3 significant digits:

$$\left[\begin{array}{rrr|r} 10 & -7 & 0 & 7 \\ -3 & 2.09 & 6 & 3.91 \\ 5 & -1 & 5 & 6 \end{array}\right] \to \left[\begin{array}{rrr|r} 10 & -7 & 0 & 7 \\ 0 & -0.01 & 6 & 6.01 \\ 0 & 2.5 & 5 & 2.5 \end{array}\right] \to \left[\begin{array}{rrr|r} 10 & -7 & 0 & 7 \\ 0 & -0.01 & 6 & 6.01 \\ 0 & 0 & 1500 & 1500 \end{array}\right]$$

By back-substitution, the solution is $x = 0.00$, $y = -1$, $z = 1$.

If we do partial pivoting, the first step remains the same but then we must switch the rows before going further. The row echelon form is

$$\left[\begin{array}{rrr|r} 10 & -7 & 0 & 7 \\ 0 & 2.5 & 5 & 2.5 \\ 0 & 0 & 6.02 & 6.02 \end{array}\right]$$

and back-substituting (and rounding) gives $x = 0.00$, $y = -1.00$, $z = 1.00$.

In this case, it doesn't matter whether we use partial pivoting or not. This is because the numbers involved in the three equations are all roughly of the same scale.

5. Write a short paragraph summarizing what you learned from this Application Mini-Project.

Solution: It doesn't come as a shock that rounding introduces errors. The possible magnitude of these errors was illustrated in this project. In the first portion of the project, we considered some ill-conditioned systems of equations, i.e., consistent systems that were within a rounding error of being inconsistent. It is important in such situations to keep rounding errors to a minimum by, for example, working with exact numbers until the very end of the problem. In the second portion of the project, we explored partial pivoting. This idea is relevant when the equations involved in the system under consideration have coefficients that differ by an order of magnitude. In that situation, we want to be careful not to lose information by rounding in such a way that one of the equations essentially becomes lost. We can do this through partial pivoting.