Problem Set 6
Due: 11:30 a.m. on Tuesday, April 9

Instructions: All students except for the presenter are to submit solutions to **two** of the exercises below. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises:

1. Let $\mathcal{H} = (\mathcal{E}_1, \ldots, \mathcal{E}_d)$ be a hypergraph on $V(\mathcal{H}) = \{x_1, \ldots, x_n\}$ such that $|\mathcal{E}_j| = 2$, for all $j = 1, \ldots, d$ (that is, $\mathcal{H}$ is a simple graph).
   (a) Using the notion of the incidence matrix of $\mathcal{H}$, give conditions that $\mathcal{H}$ must satisfy in order for its dual $\mathcal{H}^*$ to also be a simple graph.
   (b) Suppose $\mathcal{H} = (\mathcal{E}_1, \ldots, \mathcal{E}_8)$ is a hypergraph (with $|\mathcal{E}_j| = 2$, for all $j = 1, \ldots, 8$) on $V(\mathcal{H}) = \{x_1, \ldots, x_8\}$ whose incidence matrix is given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Use part (a) to determine whether $\mathcal{H}$ is a simple graph or not. Is $\mathcal{H}^*$ a simple graph? Justify your answer. Do not draw $\mathcal{H}$.

2. Let $\mathcal{H} = (\mathcal{E}_1, \ldots, \mathcal{E}_d)$ be a hypergraph on $V(\mathcal{H}) = \{x_1, \ldots, x_n\}$. Prove that the dual of a subhypergraph of $\mathcal{H}$ is a partial hypergraph of the dual hypergraph $\mathcal{H}^*$.

3. Let $\mathcal{H} = (\mathcal{E}_1, \ldots, \mathcal{E}_d)$ be a simple hypergraph on $V(\mathcal{H}) = \{x_1, \ldots, x_n\}$ satisfying the following two conditions:
   (i) $\mathcal{E}_j \cap \mathcal{E}_k \neq \emptyset$ for all $j \neq k$;
   (ii) $\mathcal{E}_j \cup \mathcal{E}_k \neq V(\mathcal{H})$ for all $j \neq k$.

   Prove that $\mathcal{H}' = (\mathcal{E}_1, \ldots, \mathcal{E}_d, V(\mathcal{H}) \setminus \mathcal{E}_1, \ldots, V(\mathcal{H}) \setminus \mathcal{E}_d)$ is a simple hypergraph on $V(\mathcal{H})$.