# Problem Set 3
## Due: 11:30 a.m. on Tuesday, March 5

*Instructions:* All students except for the presenter are to complete all of the exercises below. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets.* Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

*Exercises:*

1. Let $n \geq 2$ be an integer. Consider the ideal $I \subseteq k[x_0, x_1, \ldots, x_{2^n-1}]$ defined by

   $$I = \langle x_i x_j \mid \text{the binary representations of } i \text{ and } j \text{ differ in } \textit{at least two} \text{ digits} \rangle .$$

   Use Fröberg's Theorem to show that $I$ does *not* have a linear resolution. *Hint:* When considering the complementary graph, what are the parities of a pair of adjacent vertices?

   **Extra Credit Opportunity:** For $1 \leq m < n$ where $m$ is odd, determine if the ideal

   $$J = \langle x_i x_j \mid \text{the binary representations of } i \text{ and } j \text{ matches in } m \text{ digits} \rangle$$

   has a linear resolution. Explain your reasoning. In this case, you may assume that the associated complementary graph has a cycle.

2. (*The Short Five Lemma*) Let $R$ be a commutative ring and consider the following commutative diagram of $R$-modules and $R$-module homomorphisms where each row is an exact sequence.

   $$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

   Prove **one** of the following facts:

   (a) If $\alpha$ and $\gamma$ are injective, then $\beta$ is also injective.

   (b) If $\alpha$ and $\gamma$ are surjective, then $\beta$ is also surjective.