Problem Set 9

Due: 10:00 a.m. on Thursday, March 21

Instructions: Carefully read Sections 4.5–4.6 and 5.1 of the textbook. MATH 7340 students should submit solutions to all of the following problems and MATH 4340 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: From the textbook *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, fourth edition, by David A. Cox, John Little, and Donal O'Shea.

Throughout, k denotes a field.

1U. (Section 4.5 #6) Let k be an infinite field.

   (a) Show that any straight line in k^n is irreducible.

   (b) Prove that any vector subspace of k^n is irreducible. *Hint:* Use a parametrization.

2U. (Section 4.5 #7) Show that

$$\mathbf{I}(\{(a_1, \ldots, a_n)\}) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle.$$

3U. (Section 4.5 #8b) If $I \subseteq \mathbb{R}[x_1, \ldots, x_n]$ is a maximal ideal, show that $\mathbf{V}(I)$ is either empty or a point in $\mathbb{R}^n$. You may use without proof that the ideal $\langle x_1 - a_1, \ldots, x_n - a_n \rangle \subset \mathbb{R}[x_1, \ldots, x_n]$ is a maximal ideal.

4U. (Section 4.6 #6) Let V, W be varieties with $W \subsetneq V$. Prove that if $V \setminus W$ is Zariski dense in V then W contains no irreducible component of V. *Hint:* $V_i \subseteq W$ implies that $V \setminus W \subseteq V \setminus V_i$.

5U. (Section 5.1 #9) Let V be an irreducible variety and let ϕ, ψ be functions in $k[V]$ represented by polynomials f, g, respectively. Assume that $\phi \cdot \psi = 0$ in $k[V]$ and that neither ϕ not ψ is the zero function on V.

   (a) Show that $V = (V \cap \mathbf{V}(f)) \cup (V \cap \mathbf{V}(g))$.

   (b) Show that neither $V \cap \mathbf{V}(f)$ nor $V \cap \mathbf{V}(g)$ is all of V and deduce a contradiction.

6. (Section 5.1 #6) Consider the mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^5$ defined by $\phi(u, v) = (u, v, u^2, uv, v^2)$.

   (a) The image of ϕ is a variety S known as an affine Veronese surface. Find implicit equations for S. You may use the fact that

$$G = \{u - x_1, v - x_2, x_1^2 - x_3, x_1x_2 - x_4, x_1x_4 - x_2x_3, x_1x_5 - x_2x_4, x_2^2 - x_5, x_3x_5 - x_4^2\}$$

   is a Gröbner basis for $\langle x_1 - u, x_2 - v, x_3 - u^2, x_4 - uv, x_5 - v^2 \rangle$ with respect to lex order with $u > v > x_1 > x_2 > x_3 > x_4 > x_5$.

   (b) Show that the projection $\pi : S \to \mathbb{R}^2$ defined by $\pi(x_1, x_2, x_2, x_4, x_5) = (x_1, x_2)$ is the inverse mapping of $\phi : \mathbb{R}^2 \to S$. What does this imply about S and $\mathbb{R}^2$?