Problem Set 8
Due: 10:00 a.m. on Thursday, March 14

Instructions: Carefully read Sections 4.2–4.4 of the textbook. MATH 7340 students should submit solutions to all of the following problems and MATH 4340 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: From the textbook *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, fourth edition, by David A. Cox, John Little, and Donal O'Shea.

Throughout, k denotes a field.

1U. (Section 4.2 #5) Prove that $\mathbf{I}$ and $\mathbf{V}$ are inclusion-reversing and that $\mathbf{V}(\sqrt{I}) = \mathbf{V}(I)$ for any ideal I.

2U. (Section 4.2 #14) Let $J = \langle xy, (x-y)x\rangle$. Describe $\mathbf{V}(J)$ and show that $\sqrt{J} = \langle x\rangle$.

3U. (Section 4.3 #9) Let I and J be ideals in $R = k[x_1, \dots, x_n]$ where k is an arbitrary field.

   (a) Show that $\sqrt{IJ} = \sqrt{I \cap J}$.

   (b) Give an example to show that the product of radical ideals need not be radical.

   (c) Give an example to show that $\sqrt{IJ}$ can differ from $\sqrt{I}\sqrt{J}$.

4U. (Section 4.4 #1b) Find the Zariski closure of the boundary of the first quadrant in $\mathbb{R}^2$. That is, find the Zariski closure of the set $S = \{(a,0) \in \mathbb{R}^2 \mid a \geq 0\} \cup \{(0,b) \in \mathbb{R}^2 \mid b \geq 0\}$. *Hint:* Determine $\mathbf{I}(S)$. In doing so, it might be helpful to take $f \in \mathbf{I}(S)$ and write $f(x,y) = A(x,y)xy + B(x)x + C(y)y + D$ where $B(x) \in \mathbb{R}[x], C(y) \in \mathbb{R}[y]$ and $D \in \mathbb{R}$.

5U. (Section 4.4 #8) Let $V, W \subseteq k^n$ be varieties. Prove that $\mathbf{I}(V) : \mathbf{I}(W) = \mathbf{I}(V \setminus W)$. You may use the fact that $I : J \subseteq \mathbf{I}(\mathbf{V}(I) \setminus \mathbf{V}(J))$.

6. (Section 4.4 #4) Let I and J be ideals in $k[x_1, \dots, x_n]$. Proposition 9 of Section 4.4 says that $I : J \subseteq I : J^\infty$. Prove that if I is radical, then $I : J = I : J^\infty$.