Problem Set 7

Due: 10:00 a.m. on Thursday, March 7

Instructions: Carefully read Sections 3.6–4.1 of the textbook. MATH 7340 students should submit solutions to all of the following problems and MATH 4340 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: From the textbook *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, fourth edition, by David A. Cox, John Little, and Donal O'Shea.

Throughout, k denotes a field.

1U. (Section 3.6 #13) Let $f = x^2y + x - 1$ and $g = x^2y + x + y^2 - 4$. If $h = \text{Res}(f, g, x) \in \mathbb{C}[y]$, show that $h(0) = 0$. But if we substitute $y = 0$ into f and g, we get $x - 1$ and $x - 4$. Show that these polynomials have a non-zero resultant. Thus, $h(0)$ is *not* a resultant.

2U. (Section 3.6 #21) Suppose that $I = \langle f, g\rangle \subseteq \mathbb{C}[x, y]$ and assume that $\text{Res}(f, g, x) \neq 0$. Prove that $\mathbf{V}(I_1) = \pi_1(V)$, where $V = \mathbf{V}(I)$ and π_1 is projection onto the y-axis.

3U. (Section 4.1 #1) Recall that $\mathbf{V}(y - x^2, z - x^3)$ is the twisted cubic in $\mathbb{R}^3$.

(a) Show that $\mathbf{V}((y - x^2)^2 + (z - x^3)^2)$ is also the twisted cubic.

(b) Show that any variety $\mathbf{V}(I) \subseteq \mathbb{R}^n, I \subseteq \mathbb{R}[x_1, \ldots, x_n]$, can be defined by a single equation (and hence by a principal ideal).

4U. (Section 4.1 #2) Let $J = \langle x^2 + y^2 - 1, y - 1\rangle$. Find $f \in \mathbf{I}(\mathbf{V}(J))$ such that $f \notin J$. You may use the fact that $G = \{x^2, y - 1\}$ is a Gröbner basis for J with respect to lex order with $x > y$.

5. (Section 4.1 #10) In Exercise 3U., you encountered two ideals in $\mathbb{R}[x, y]$ that give the same non-empty variety. Show that one of these ideals is contained in the other. Can you find two ideals in $\mathbb{R}[x, y]$, neither contained in the other, which give the same non-empty variety? Can you do the same for $\mathbb{R}[x]$?