Problem Set 6
Due: 10:00 a.m. on Thursday, February 28

Instructions: Carefully read Sections 3.2–3.6 of the textbook. MATH 7340 students should submit solutions to all of the following problems and MATH 4340 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: From the textbook *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, fourth edition, by David A. Cox, John Little, and Donal O'Shea.

Throughout, k denotes a field.

1U. (Section 3.2 #4) To see how the Closure Theorem can fail over $\mathbb{R}$, consider the ideal

$$I = \langle x^2 + y^2 + z^2 + 2, 3x^2 + 4y^2 + 4z^2 + 5 \rangle.$$

Let $V = \mathbf{V}(I)$, and let π_1 be the projection taking (x, y, z) to (y, z). You may use the fact that $G = \{x^2 + 3, y^2 + z^2 - 1\}$ is a Gröbner basis for I with respect to lex with $x > y > z$.

(a) Working over $\mathbb{C}$, prove that $\mathbf{V}(I_1) = \pi_1(V)$.

(b) Working over $\mathbb{R}$, prove that $V = \emptyset$ and that $\mathbf{V}(I_1)$ is infinite. Thus, $\mathbf{V}(I_1)$ may be much larger than the smallest variety containing $\pi_1(V)$ when the field is not algebraically closed.

2U. (Section 3.2 #5) Suppose that $I \subseteq \mathbb{C}[x, y]$ is an ideal such that $I_1 \neq \{0\}$. Prove that $\mathbf{V}(I_1) = \pi_1(V)$, where $V = \mathbf{V}(I)$ and π_1 is the projection onto the y-axis. *Hint:* Use part (i) of the Closure Theorem. Also, the only varieties contained in $\mathbb{C}$ are either $\mathbb{C}$ or finite subsets of $\mathbb{C}$.

3U. (Section 3.3 #7) Let S be the parametric surface

$$\begin{aligned} x &= uv, \\ y &= uv^2, \\ z &= u^2. \end{aligned}$$

(a) Find the equation of the smallest variety V that contains S. You may use the fact that a Gröbner basis G for the ideal $I = \langle x - uv, y - uv^2, z - u^2 \rangle$ with respect to lex order with $u > v > x > y > z$ is $G = \{g_1, \ldots, g_8\}$, where

$$\begin{aligned} g_1 &= u^2 - z, \\ g_2 &= uv - x, \\ g_3 &= ux - vz, \\ g_4 &= uy - x^2, \\ g_5 &= v^2 z - x^2, \\ g_6 &= vx - y, \\ g_7 &= vyz - x^3, \\ g_8 &= x^4 - y^2 z. \end{aligned}$$

(b) Over $\mathbb{C}$, show that V contains points which are not in S. Determine exactly which points of V are not on S.

4U. (Section 3.4 #8) In this problem, we will compute some singular points. For both parts, work over $\mathbb{R}$.

(a) Exercise 8 of Section 1.3 studies the curve $y^2 = cx^2 - x^3$, where c is some constant. Find all singular points of this curve and explain how your answer relates to the picture of the curve given on page 24 of the textbook.

(b) Show that the circle $x^2 + y^2 = a^2$ in $\mathbb{R}^2$ has no singular points when $a > 0$.

5U. (Section 3.6 #1) Compute the resultant of $x^5 - 3x^4 - 2x^3 + 7x + 6$ and $x^4 + x^2 + 1$. Do these polynomials have a common factor in $\mathbb{Q}[x]$? Explain your reasoning.

6. (Section 3.4 #1) Let C be the curve in k^2 defined by $x^2 - xy + y^2 = 1$ and note that $(1, 1) \in C$. Now consider the straight line parametrized by

$$x = 1 + ct,$$
$$y = 1 + dt.$$

Compute the multiplicity of this line when it meets C at $(1, 1)$. What does this tell you about the tangent line? *Hint:* There will be two cases to consider.