# Problem Set 5
## Due: 10:00 a.m. on Thursday, February 14

*Instructions:* Carefully read Sections 2.9–3.2 of the textbook. MATH 7340 students should submit solutions to all of the following problems and MATH 4340 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

*Exercises:* From the textbook *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, fourth edition, by David A. Cox, John Little, and Donal O'Shea.

Throughout, $k$ denotes a field.

1U. (Section 2.9 #2) Consider the ideal $I = \langle x^2+y+z-1, x+y^2+z-1, x+y+z^2-1\rangle \subseteq \mathbb{Q}[x,y,z]$.

(a) Show that the generators of $I$ fail to be a Gröbner basis for any lex order.

(b) Find a monomial order for which the leading terms of the generators are relatively prime.

(c) Explain why the generators automatically form a Gröbner basis for the monomial order you found in part (b).

2U. (Section 3.1 #1) Let $I \subseteq k[x_1,\dots,x_n]$ be an ideal.

(a) Prove that $I_l = I \cap k[x_{l+1},\dots,x_n]$ is an ideal of $k[x_{l+1},\dots,x_n]$.

(b) Prove that the ideal $I_{l+1} \subseteq k[x_{l+2},\dots,x_n]$ is the first elimination ideal of $I_l \subseteq k[x_{l+1},\dots,x_n]$. This observation allows us to use the Extension Theorem multiple times when eliminating more than one variable.

3U. (Section 3.1 #4) Consider the equations:

$$\begin{aligned} x^2+y^2+z^2 &= 4 \\ x^2+2y^2 &= 5 \\ xz &= 1. \end{aligned}$$

(a) Let $I = \langle x^2+y^2+z^2-4, x^2+2y^2-5, xz-1\rangle$. A Gröbner basis for $I$ with respect to the lex ordering with $x>y>z$ is:

$$G = \{x+2z^3-3z, y^2-z^2-1, 2z^4-3z^2+1\}.$$

Find bases for the elimination ideals $I_1$ and $I_2$.

(b) How many rational solutions are there? Support your answer by showing work for the Extension Steps involved.

4U. (Section 3.1 #9) Consider the system of equations given by

$$\begin{aligned} x^5+\frac{1}{x^5} &= y \\ x+\frac{1}{x} &= z. \end{aligned}$$

Let $I$ be the ideal in $\mathbb{C}[x,y,z]$ determined by these equations. By clearing denominators, you can work with the ideal $I = \langle x^{10}+1-x^5y, x^2+1-xz\rangle$ which has Gröbner basis $G = \{x^2-xz+1, y-z^5+5z^3-5z\}$ with respect to the lex order with $x>y>z$.

(a) Find a basis of $I_1 \subseteq \mathbb{C}[y,z]$ and show that $I_2 = \{0\}$.

(b) Use the Extension Theorem to prove that each partial solution $c \in \mathbf{V}(I_2) = \mathbb{C}$ extends to a solution in $\mathbf{V}(I) \subseteq \mathbb{C}^3$.

(c) Which partial solutions $(b, c) \in \mathbf{V}(I_1) \subseteq \mathbb{R}^2$ extend to solutions in $\mathbf{V}(I) \subseteq \mathbb{R}^3$? Explain why your answer does not contradict the Extension Theorem.

(d) If we regard $z$ as a "parameter", then solve for $x$ and $y$ as algebraic functions of $z$ to obtain a "parametrization" of $\mathbf{V}(I)$.

5. (Section 3.2 #1) Prove the Geometric Extension Theorem using the Extension Theorem and Lemma 1 of Section 3.2.