Problem Set 4
Due: 10:00 a.m. on Thursday, February 7

Instructions: Carefully read Sections 2.6–2.8 of the textbook. MATH 7340 students should submit solutions to all of the following problems and MATH 4340 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets.* Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: From the textbook *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, fourth edition, by David A. Cox, John Little, and Donal O'Shea.

Throughout, k denotes a field.

1U. (Section 2.6 #4) Let G and G' be Gröbner bases for a non-zero ideal I with respect to the same monomial order in $k[x_1, \dots, x_n]$. Show that $\overline{f}^G = \overline{f}^{G'}$ for all $f \in k[x_1, \dots, x_n]$. Hence, the remainder on division by a Gröbner basis is even independent of which Gröbner basis we use, as long as we use one particular monomial order. *Hint:* See Exercise 1 of Section 2.6 in the textbook.

2U. (Section 2.6 #10) Using Buchberger's Criterion, determine whether the following sets G are Gröbner bases for the ideal they generate.

(a) $G = \{x^2 - y, x^3 - z\}$ for grlex order.
(b) $G = \{x^2 - y, x^3 - z\}$ for invlex order.

3U. (Section 2.6 #13) Let $I \subseteq k[x_1, \dots, x_n]$ be a non-zero ideal, and let G be a Gröbner basis of I.

(a) Show that $\overline{f}^G = \overline{g}^G$ if and only if $f - g \in I$. *Hint:* See Exercise 1 of Section 2.6 in the textbook.
(b) Use Exercise 1 of Section 2.6 to show that
$$\overline{f+g}^G = \overline{f}^G + \overline{g}^G.$$
(c) Deduce that
$$\overline{fg}^G = \overline{\overline{f}^G \cdot \overline{g}^G}^G.$$

4U. (Section 2.7 #2 & 3) Let $I = \langle x^2 + y, x^4 + 2x^2y + y^2 + 3\rangle$.

(a) Use Buchberger's Algorithm to find a Gröbner basis for I with respect to the lex order with $x > y$.
(b) What does you result to part (a) indicate about the variety $\mathbf{V}(I)$?
(c) Find a reduced Gröbner basis for I with respect to the lex order with $x > y$.

5. (Section 2.8 #7) Let the surface S in $\mathbb{R}^3$ be formed by taking the *union* of the straight lines joining pairs of points on the lines
$$\{x = t, y = 0, z = 1\} \quad \text{and} \quad \{x = 0, y = 1, z = t\}$$
with the *same parameter value* (i.e., the same t). (This is a special example of a class of surfaces called *ruled surfaces.*)

(a) Show that the surface S can be given parametrically as
$$x = ut,$$
$$y = 1 - u,$$
$$z = t + u(1 - t).$$
Hint: The line joining the points $P = (t, 0, 1)$ and $Q = (0, 1, t)$ is parametrized by $(x, y, z) = uP + (1 - u)Q$.

(b) The set $G = \{1 - 2y + xy + y^2 - z + yz, -1 + u + y, 1 + t - x - y - z\}$ is a Gröbner basis for the ideal $\langle x - tu, y - 1 + u, z - t - u + tu \rangle$ with respect to the lex order with $t > u > x > y > z$. Using the method of Examples 4 and 5 from Section 2.8 in the textbook, find an (implicit) equation of a variety V containing the surface S.

(c) Show $V = S$ (that is, show that every point of the variety V can be obtained by substituting some values for t, u in the equations of part (a). *Hint:* Try to "solve" the implicit equation of V for one variable as a function of the other two.