Problem Set 3
Due: 10:00 a.m. on Thursday, January 31

Instructions: Carefully read Sections 2.4–2.5 of the textbook. MATH 7340 students should submit solutions to all of the following problems and MATH 4340 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: From the textbook *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, fourth edition, by David A. Cox, John Little, and Donal O'Shea.

Throughout, k denotes a field.

1U. (Section 2.4 #3) Let $I = \langle x^6, x^2y^3, xy^7\rangle \subseteq k[x, y]$.

(a) In the (m, n)-plane, plot the set of exponent vectors (m, n) of monomials x^my^n appearing in elements of I.

(b) If we apply the Division Algorithm to an element $f \in k[x, y]$, using the generators of I as divisors, what terms can appear in the remainder?

2U. (Section 2.4 #5) Suppose that $I = \langle x^\alpha \mid \alpha \in A\rangle$ is a monomial ideal, and let S be the set of all exponents that occur as monomials of I. For any monomial order $>$, prove that the smallest element of S with respect to $>$ must lie in A.

3U. (Section 2.4 #8) If $I = \langle x^{\alpha(1)}, \ldots, x^{\alpha(s)}\rangle$ is a monomial ideal, prove that a polynomial f is in I if and only if the remainder of f on division by $x^{\alpha(1)}, \ldots, x^{\alpha(s)}$ is zero.

4U. (Section 2.5 #5) Let I be an ideal of $k[x_1 \ldots, x_n]$. Show that $G = \{g_1, \ldots, g_t\} \subseteq I$ is a Gröbner basis of I if and only if the leading term of any element of I is divisible by one of the $\mathrm{LT}(g_i)$.

5U. (Section 2.5 #13) Let

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$$

be a **descending chain** of affine varieties. Show that there is some $N \geq 1$ such that $V_N = V_{N+1} = V_{N+2} = \cdots$.

6. (Section 2.4 #7) Prove that Dickson's lemma is equivalent to the following statement: given a nonempty subset $A \subseteq \mathbb{Z}^n_{\geq 0}$, there are finitely many elements $\alpha(1), \ldots, \alpha(s) \in A$ such that for every $\alpha \in A$, there exists some i and some $\gamma \in \mathbb{Z}^n_{\geq 0}$ such that $\alpha = \alpha(i) + \gamma$.

7. (Section 2.5 #15) Given polynomials $f_1, f_2, \ldots \in k[x_1, \ldots, x_n]$, let $\mathbf{V}(f_1, f_2, \ldots) \subseteq k^n$ be the solutions of the infinite system of equations $f_1 = f_2 = \cdots = 0$. Show that there is some N such that $\mathbf{V}(f_1, f_2, \ldots) = \mathbf{V}(f_1, \ldots, f_N)$.

8. (Section 2.5 #18) When an ideal has a basis where some of the elements can be factored, we can use the factorization to help understand the variety.

(a) Show that if $g \in k[x_1, \ldots, x_n]$ factors as $g = g_1g_2$, then for any f, we have

$$\mathbf{V}(f, g) = \mathbf{V}(f, g_1) \cup \mathbf{V}(f, g_2).$$

(b) Show that in $\mathbb{R}^3$, $\mathbf{V}(y - x^2, xz - y^2) = \mathbf{V}(y - x^2, xz - x^4)$.

(c) Use part (a) to describe the variety from part (b).