Problem Set 2

Due: 10:00 a.m. on Thursday, January 24

Instructions: Carefully read Sections 1.4–1.5 and 2.1–2.3 of the textbook. MATH 7340 students should submit solutions to all of the following problems and MATH 4340 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: From the textbook *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, fourth edition, by David A. Cox, John Little, and Donal O'Shea.

Throughout, k denotes a field.

1U. (Section 1.4 #5) Show that $\mathbf{V}(x + xy, y + xy, x^2, y^2) = \mathbf{V}(x, y)$.

2U. (Section 1.4 #8) The ideal $\mathbf{I}(V)$ of a variety has a special property not shared by all ideals. Specifically, we define an ideal I to be *radical* if whenever a power f^m of a polynomial f is in I, then f itself is in I. More succinctly, I is radical when $f \in I$ if and only if $f^m \in I$ for some positive integer m.

   (a) Prove that $\mathbf{I}(V)$ is always a radical ideal.

   (b) Prove that $\langle x^2, y^2\rangle$ is not a radical ideal. This implies that $\langle x^2, y^2\rangle \neq \mathbf{I}(V)$ for any variety $V \subseteq k^2$.

3U. (Section 1.5 #11) In this exercise we will study the one-variable case of the *consistency problem* from Section 1.2. Given $f_1, \dots, f_s \in k[x]$, this asks if there is an algorithm to decide whether $\mathbf{V}(f_1, \dots, f_s)$ is non-empty. We will see that the answer is yes when $k = \mathbb{C}$.

   (a) Let $f \in \mathbb{C}[x]$ be a non-zero polynomial. Then use Theorem 7 of Section 1.1 to show that $\mathbf{V}(f) = \emptyset$ if and only if f is constant.

   (b) If $f_1, \dots, f_s \in \mathbb{C}[x]$, prove $\mathbf{V}(f_1, \dots, f_s) = \emptyset$ if and only if $\gcd(f_1, \dots, f_s) = 1$.

   (c) Describe (in words, not pseudo-code) an algorithm for determining whether or not $\mathbf{V}(f_1, \dots, f_s)$ is non-empty.

4U. (Section 2.2 #2) Each of the following polynomials is written with its monomials ordered according to (exactly) one of lex, grlex, or grevlex order. Determine which monomial order was used in each case.

   (a) $f(x, y, z) = 7x^2y^4z - 2xy^6 + x^2y^2$

   (b) $f(x, y, z) = xy^3z + xy^2z^2 + x^2z^3$

   (c) $f(x, y, z) = x^4y^5z + 2x^3y^2z - 4xy^2z^4$

5U. (Section 2.2 #6) Another monomial order is the **inverse lexicographic** or **invlex** order defined by the following: for $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, $\alpha >_{invlex} \beta$ if and only if the rightmost non-zero entry of $\alpha - \beta$ is positive. Show that invlex is equivalent to the lex order with the variables permuted in a certain way. (Which permutation?)

6U. (Section 2.2 #7) Let $>$ be any monomial order.

(a) Show that $\alpha \geq 0$ for all $\alpha \in \mathbb{Z}^n_{\geq 0}$. *Hint:* Proof by contradiction.

(b) Show that if x^α divides x^β, then $\alpha \leq \beta$. Is the converse true?

(c) Show that if $\alpha \in \mathbb{Z}^n_{\geq 0}$, then α is the smallest element of $\alpha + \mathbb{Z}^n_{\geq 0} = \{\alpha + \beta \mid \beta \in \mathbb{Z}^n_{\geq 0}\}$.

7U. (Section 2.3 #5) We will study the division of $f = x^3 - x^2y - x^2z + x$ by $f_1 = x^2y - z$ and $f_2 = xy - 1$.

(a) Compute using grlex order:

$$r_1 = \text{remainder of } f \text{ on division by } (f_1, f_2).$$

$$r_2 = \text{remainder of } f \text{ on division by } (f_2, f_1).$$

Your results should be *different*. Where in the division algorithm did the difference occur?

(b) Is $r = r_1 - r_2$ in the ideal $\langle f_1, f_2 \rangle$? If so, find an explicit expression $r = Af_1 + Bf_2$. If not, say why not.

(c) Compute the remainder of r on division by (f_1, f_2). Why could you gave predicted your answer before doing the division?

(d) Find another polynomial $g \in \langle f_1, f_2 \rangle$ such that the remainder on division of g by (f_1, f_2) is non-zero. *Hint:* $(xy + 1) \cdot f_2 = x^2y^2 - 1$, whereas $y \cdot f_1 = x^2y^2 - yz$.

(e) Does the division algorithm give us a solution for the ideal membership problem for the ideal $\langle f_1, f_2 \rangle$? Explain your answer.

8 (Section 2.2 #10) In $\mathbb{Z}_{\geq 0}$ with the usual ordering, between any two integers, there are only a finite number of other integers. Is this necessarily true in $\mathbb{Z}^n_{\geq 0}$ for a monomial order? Is it true for the grlex order?

9 (Section 2.3 #9) The discussion around equation (2) of Section 1.4 shows that every polynomial $f \in \mathbb{R}[x, y, z]$ can be written as

$$f = h_1(y - x^2) + h_2(z - x^3) + r,$$

where r is a polynomial in x alone and $\mathbf{V}(y - x^2, z - x^3)$ is the twisted cubic curve in $\mathbb{R}^3$.

(a) Give a proof of this fact using the division algorithm. *Hint:* You need to specify carefully the monomial ordering to be used.

(b) Use the parametrization of the twisted cubic to show that $z^2 - x^4y$ vanishes at every point of the twisted cubic.

(c) Find an explicit representation

$$z^2 - x^4y = h_1(y - x^2) + h_2(z - x^3)$$

using the division algorithm.