Problem Set 11
Due: 10:00 a.m. on Thursday, April 4

Instructions: Carefully read Sections 8.1–8.2 of the textbook. MATH 7340 students should submit solutions to all of the following problems and MATH 4340 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: From the textbook *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, fourth edition, by David A. Cox, John Little, and Donal O'Shea.

Throughout, k denotes a field.

1U. (Section 8.1 #9) We can view parts of $\mathbb{P}^2(\mathbb{R})$ in the (x, y) and (x, z) coordinate systems. In the (x, z) picture, it is natural to ask what happened to y. To see this, we will study how (x, y) coordinates look when viewed in the plane with (x, z) coordinates.

(a) Show that (a, b) in the (x, y)-plane gives the point $(a/b, 1/b)$ in the (x, z)-plane.

(b) Use the formula of part (a) to study what the parabolas $(x, y) = (t, t^2)$ and $(x, y) = (t^2, t)$ look like in the (x, z)-plane. Draw pictures of what happens in both (x, y) and (x, z) coordinates.

2U. (Section 8.2 #6) This problem studies the subsets $U_i \subseteq \mathbb{P}^n(k)$ defined in class.

(a) In $\mathbb{P}^4(k)$, identify the points that are in the subsets $U_2, U_2 \cap U_3$, and $U_1 \cap U_3 \cap U_4$.

(b) Give an identification of $\mathbb{P}^4(k) \setminus U_2, \mathbb{P}^4(k) \setminus (U_2 \cup U_3)$, and $\mathbb{P}^4(k) \setminus (U_1 \cup U_3 \cup U_4)$ as a "copy" of another projective space.

(c) In $\mathbb{P}^4(k)$, which points are $\cap_{i=0}^4 U_i$?

(d) In general, describe the subset $U_{i_1} \cap \cdots \cap U_{i_s} \subseteq \mathbb{P}^n(k)$, where

$$1 \leq i_1 < i_2 < \cdots < i_s \leq n.$$

3U. (Section 8.2 #8b) By dehomogenizing the defining equations of the projective variety

$$V = \mathbf{V}(x_0x_2 - x_3x_4, x_0^2x_3 - x_1x_2^2) \subseteq \mathbb{P}^4(k),$$

find equations for the affine variety $V \cap U_0 \subseteq k^4$. Do the same for $V \cap U_3$.

4U. (Section 8.2 #14a) Let W be the affine variety $W = \mathbf{V}(y^2 - x^3 - ax - b) \subseteq \mathbb{R}^2, a, b \in \mathbb{R}$.

(a) Apply the homogenization process to write $W = V \cap U_0$, where V is a projective variety.

(b) Identify $V \setminus W = V \cap H$, where H is the hyperplane at infinity.

(c) Is the point $V \cap H$ singular?

Hint: Let the homogeneous coordinates on $\mathbb{P}^2(\mathbb{R})$ be $(z : x : y)$ so that U_0 is where $z \neq 0$.

5. (Section 8.2 #21a) When we have a curve parametrized by $t \in k$, there are many situations where we want to let $t \to \infty$. Since $\mathbb{P}^1(k) = k \cup \{\infty\}$, this suggests that we should let our parameter space be $\mathbb{P}^1(k)$. Here is an example of how this works: Consider the parametrization

$$(x, y) = \left(\frac{1 + t^2}{1 - t^2}, \frac{2t}{1 - t^2}\right)$$

of the hyperbola $x^2 - y^2 = 1$ in $\mathbb{R}^2$. To make this projective, we first work in $\mathbb{P}^2(\mathbb{R})$ and write the parametrization as

$$\left(\frac{1+t^2}{1-t^2} : \frac{2t}{1-t^2} : 1\right) = (1+t^2 : 2t : 1-t^2).$$

The next step is to make t projective. Given $(a : b) \in \mathbb{P}^1(\mathbb{R})$, we can write it as $(1 : t) = (1 : b/a)$ provided $a \neq 0$. Now substitute $t = b/a$ into the right-hand side and clear denominators. Explain why this gives a well-defined map $\mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$.