Problem Set 10
Due: 10:00 a.m. on Thursday, March 28

Instructions: Carefully read Sections 5.4–5.5 of the textbook. MATH 7340 students should submit solutions to all of the following problems and MATH 4340 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: From the textbook *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, fourth edition, by David A. Cox, John Little, and Donal O'Shea.

Throughout, k denotes a field.

1U. (Section 5.4 #1) Let C be the twisted cubic curve in k^3.

(a) Show that C is a subvariety of the surface $S = \mathbf{V}(xz - y^2)$.

(b) Find an ideal $J \subseteq k[S]$ such that $C = \mathbf{V}_S(J)$.

2U. (Section 5.4 #9) Let $\alpha : V \to W$ and $\beta : W \to V$ be inverse polynomial mappings between two isomorphic varieties V and W. Let $U = \mathbf{V}_V(I)$ for some ideal $I \subseteq k[V]$.

(a) Define $J = \{\beta^*(\phi) \mid \phi \in I\}$. Prove that J is an ideal of $k[W]$.

(b) Show that $\alpha(U)$ is a subvariety of W by proving that $\alpha(U) = \mathbf{V}_W(J)$.

3U. (Section 5.5 #7) Let $S = \mathbf{V}(x^2 + y^2 + z^2 - 1)$ in $\mathbb{R}^3$ and let $W = \mathbf{V}(z)$ be the (x, y)-plane. In this exercise, we will show that S and W are birationally equivalent varieties, via an explicit mapping called *stereographic projection*.

(a) Derive parametric equations as we did in our example from class on March 21 for the line L_q in $\mathbb{R}^3$ passing through the north pole $(0, 0, 1)$ of S and a general point $q = (x_0, y_0, z_0) \neq (0, 0, 1)$ in S.

(b) Using the line from part (a) show that $\phi(q) = L_q \cap W$ defines a rational mapping $\phi : S \dashrightarrow \mathbb{R}^2$ This mapping is the stereographic projection.

(c) Show that the rational parametrization of S given by

$$x = \frac{2u}{u^2 + v^2 + 1}$$
$$y = \frac{2v}{u^2 + v^2 + 1}$$
$$z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

is the inverse mapping, denoted ψ, of ϕ. Be sure to explain where $\phi \circ \psi$ and $\psi \circ \phi$ are defined.

(d) Deduce that S and W are birationally equivalent varieties and find subvarieties $S' \subseteq S$ and $W' \subseteq W$ such that ϕ and ψ put $S \setminus S'$ and $W \setminus W'$ into one-to-one correspondence.

4. (Section 5.4 #16) Let k be algebraically closed. Prove the Weak Nullstellensatz for $k[V]$, which asserts that for any ideal $J \subseteq k[V]$, $\mathbf{V}_V(J) = \emptyset$ if and only if $J = k[V]$.