Problem Set 1
Due: 10:00 a.m. on Thursday, January 17

Instructions: Carefully read Sections 1.1–1.3 of the textbook. MATH 7340 students should submit solutions to all of the following problems and MATH 4340 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: From the textbook *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, fourth edition, by David A. Cox, John Little, and Donal O'Shea.

Throughout, k denotes a field.

1U. (Section 1.1 #2) Let $\mathbb{F}_2$ be the field $\mathbb{F}_2 = \{0, 1\}$.

(a) Consider the polynomial $g(x, y) = x^2y+y^2x \in \mathbb{F}_2[x, y]$. Show that $g(x, y) = 0$ for every $(x, y) \in \mathbb{F}_2^2$, and explain why this does not contradict Proposition 5.

(b) Find a non-zero polynomial in $\mathbb{F}_2[x, y, z]$ which vanishes at every point of $\mathbb{F}_2^3$. Try to find one involving all three variables.

(c) Find a non-zero polynomial in $\mathbb{F}_2[x_1, \dots, x_n]$ which vanishes at every point of $\mathbb{F}_2^n$. Can you find one in which all of $x_1, \dots, x_n$ appear?

2U. (Section 1.1 #5) In the proof of Proposition 5, we took $f \in k[x_1, \dots, x_n]$ and wrote it as a polynomial in x_n with coefficients in $k[x_1, \dots, x_{n-1}]$. To see what this looks like in a specific case, consider the polynomial

$$f(x, y, z) = x^5y^2z - x^4y^3 + y^5 + x^2z - y^3z + xy + 2x - 5z + 3.$$

(a) Write f as a polynomial in x with coefficients in $k[y, z]$.

(b) Write f as a polynomial in y with coefficients in $k[x, z]$.

(c) Write f as a polynomial in z with coefficients in $k[x, y]$.

3U. (Section 1.2 #3) In the plane $\mathbb{R}^2$, draw a picture to illustrate

$$\mathbf{V}(x^2 + y^2 - 4) \cap \mathbf{V}(xy - 1) = \mathbf{V}(x^2 + y^2 - 4, xy - 1)$$

and determine the points of intersection. Note that this is a special case of Lemma 2.

4U. (Section 1.2 #6) Let us show that all finite subsets of k^n are affine varieties.

(a) Prove that a single point $(a_1, \dots, a_n) \in k^n$ is an affine variety.

(b) Prove that every finite subset of k^n is an affine variety.

5U. (Section 1.2 #8) It can take some work to show that something is *not* an affine variety. For example, consider the set

$$X = \{(x, x) \mid x \in \mathbb{R}, x \neq 1\} \subseteq \mathbb{R}^2,$$

which is the straight line $x = y$ with the point $(1, 1)$ removed. To show that X is not an affine variety, suppose that $X = \mathbf{V}(f_1, \dots, f_s)$. Then each f_i vanishes on X, and if we can show that f_i also vanishes at $(1, 1)$, we will get the desired contradiction. Thus, here is what you are to prove: if $f \in \mathbb{R}[x, y]$ vanishes on X, then $f(1, 1) = 0$. *Hint:* Let $g(t) = f(t, t)$, which is a polynomial in $\mathbb{R}[t]$. Now apply the proof of Proposition 5 in Section 1.

6U. (Section 1.2 #15) In Lemma 2 we showed that if V and W are affine varieties, then so are their union $V \cup W$ and intersection $V \cap W$. In this exercise we will study how other set-theoretic operations affect affine varieties.

(a) Prove that finite unions and intersections of affine varieties are again affine varieties.

(b) Give an example to show that an infinite union of affine varieties need not be an affine variety. Surprisingly, an infinite intersection of affine varieties is still an affine variety. This is a consequence of the Hilbert Basis Theorem, which will be discussed in Chapters 2 and 4.

(c) Give an example to show that the set-theoretic difference $V \setminus W$ of two affine varieties need not be an affine variety.

(d) Let $V \subseteq k^n$ and $W \subseteq k^m$ be two affine varieties, and let

$$V \times W = \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in k^{n+m} \mid (x_1, \ldots, x_n) \in V, (y_1, \ldots, y_m) \in W\}$$

be their Cartesian product. Prove that $V \times W$ is an affine variety in k^{n+m}. *Hint:* If V is defined by $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, then we can regard $f_1, \ldots, f_s$ as polynomials in $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$, and similarly for W. Show this gives defining equations for the Cartesian product.

7U. (Section 1.3 #4) Consider the parametric representation

$$x = \frac{t}{1+t}$$
$$y = 1 - \frac{1}{t^2}.$$

(a) Find the equation of the affine variety determined by the above parametric equations.

(b) Show that the above equations parametrize all points of the variety found in part (a) except for the point (1,1).

8. (Section 1.3 # 8) Consider the curve defined by $y^2 = cx^2 - x^3$, where c is some constant. A picture can be found on page 24 of the text when $c > 0$. Our goal is to parametrize this curve.

(a) Show that a line will meet this curve at either 0, 1, 2, or 3 points. Illustrate your answer with a picture. *Hint:* Let the equation of the line be either $x = a$ or $y = mx + b$.

(b) Show that a nonvertical line through the origin meets the curve at exactly one other point when $m^2 \neq c$. Draw a picture to illustrate this, and see if you can come up with an intuitive explanation as to why this happens.

(c) Now draw the vertical line $x = 1$. Given a point $(1, t)$ on this line, draw the line connecting $(1, t)$ to the origin. This will intersect the curve in a point (x, y). Draw a picture to illustrate this, and argue geometrically that this gives a parametrization of the entire curve.

(d) Show that the geometric description from part (c) leads to the parametrization

$$x = c - t^2$$
$$y = t(c - t^2).$$

9. (Section 1.2 #11) So far, we have discussed varieties over $\mathbb{R}$ or $\mathbb{C}$. It is also possible to consider varieties over the field $\mathbb{Q}$, although the questions here tend to be *much* harder. For example, let n be a positive integer, and consider the variety $F_n \subseteq \mathbb{Q}^2$ defined by

$$x^n + y^n = 1.$$

Notice that there are some obvious solutions when x or y is zero. We call these *trivial solutions*. An interesting question is whether or not there are any nontrivial solutions.

(a) Show that F_n has two trivial solutions if n is odd and four trivial solutions if n is even.

(b) Show that F_n has a nontrivial solution for some $n \geq 3$ if and only if Fermat's Last Theorem were false. *Fermat's Last Theorem* states that, for $n \geq 3$, the equation

$$x^n + y^n = z^n$$

has no solutions where x, y, and z are non-zero integers. The general case of this conjecture was proved by Andrew Wiles in 1994 using some very sophisticated number theory. The proof is *extremely* difficult.