Quiz 9
Sample Solutions

Name: ____________________

Student Number: ____________________

In the space provided, please write your solutions to the following exercises. *Fully explain your reasoning.* Remember to use good notation and full sentences.

Good Luck!

1. Let $\varphi : G \to H$ be a group homomorphism.

   (i) Complete the following definition: The *kernel* of φ is

   Solution: $\varphi^{-1}(\{e_H\}) = \{g \in G \mid \varphi(g) = e_H\}$.

   (ii) The map $\varphi : \mathbb{M}_2(\mathbb{R}) \to \mathbb{R}$ defined by

   $$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = b,$$

   where $\mathbb{M}_2(\mathbb{R})$ is the additive group of 2×2 matrices with entries in $\mathbb{R}$, is a group homomorphism. Determine the kernel of φ.

   Solution: We have

   $$\operatorname{Ker}(\varphi) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{R}) : b = 0\right\} = \left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a, c, d \in \mathbb{R}\right\}.$$

 Turn over for Exercises (2) and (3) ...

2. Determine (with justification!) if the map $\varphi : GL_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d$$

is a group homomorphism.

Solution: φ is not a group homomorphism since it does not preserve the group operation. For example,

$$\varphi\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 1 & 4 \\ 3 & 10 \end{pmatrix}\right) = 1 + 10 = 11$$

but

$$\varphi\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right) + \varphi\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right) = 5 + 2 = 7.$$

Since $11 \neq 7$, we have that φ does not preserve the group operation.

3. If $\varphi : G \to H$ is a group homomorphism and G is cyclic, prove that $\varphi(G)$ is also cyclic.

Solution: Say that $G = \langle g \rangle$. We claim that $\varphi(G) = \langle \varphi(g) \rangle$. Clearly $\langle \varphi(g) \rangle \subset \varphi(G)$ since for any integer n we have

$$[\varphi(g)]^n = \varphi(g^n) \in \varphi(G).$$

On the other hand, let $x \in \varphi(G)$. Then there exists an element a in G such that $x = \varphi(a)$. But $G = \langle g \rangle$ implies that $a = g^m$ for some integer m. Hence,

$$x = \varphi(a) = \varphi(g^m) = [\varphi(g)]^m \in \langle \varphi(g) \rangle$$

and so $\varphi(G) \subset \langle \varphi(g) \rangle$.