Quiz 8
Sample Solutions

Name: ______________________________

Student Number: ______________________________

In the space provided, please write your solutions to the following exercises. *Fully explain your reasoning.* Remember to use good notation and full sentences.

Good Luck!

1. Let G be a group.

   (i) Complete the following definition: A subgroup H of G is called *normal* if

   Solution: $gH = Hg$ for all $g \in G$.

   (ii) Let H and K be normal subgroups of G. Show that the intersection $N = H \cap K$ is a normal subgroup of G.

   Solution: Let $g \in G$. For any $n \in N$ we have that

$$gng^{-1} \in H$$

   and

$$gng^{-1} \in K$$

   since both H and K are normal subgroups of G. Thus,

$$gng^{-1} \in N = H \cap K.$$

   We conclude that $gNg^{-1} \subset N$ for all $g \in G$. Hence, N is a normal subgroup of G.

 Turn over for Exercises (2) and (3) ...

2. If G is an abelian group with normal subgroup H, show that G/H must also be abelian.

   Solution: Let aH and bH be elements of G/H. Then

   $$(aH)(bH) = abH = baH = (bH)(aH).$$

   Hence, G/H is abelian.

3. Prove or disprove: if H is a normal subgroup of G such that H and G/H are abelian, then G is abelian.

   Solution: This statement is false. For a counter-example, let $G = S_3$ and $H = \langle(1\ 3\ 2)\rangle = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$. We saw in class that H is a normal subgroup of G and that $G/H \cong \mathbb{Z}_2$. Thus, both H and G/H are abelian (since both groups are cyclic). However, G is not abelian since, for example,

   $$(1\ 2)(1\ 2\ 3) = (2\ 3) \neq (1\ 3) = (1\ 2\ 3)(1\ 2).$$