MATH 2020: Algebra 1 **Dr. S. Cooper, Winter 2018**

Quiz 7
Sample Solutions

Name: ________________________________

Student Number: ________________________________

In the space provided, please write your solutions to the following exercises. *Fully explain your reasoning.* Remember to use good notation and full sentences.

Good Luck!

1. Let

$$H = \left\{ A \in GL_2(\mathbb{R}) : A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right\},$$

which is a subgroup of $GL_2(\mathbb{R})$. Prove that $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is isomorphic to H.

Solution: Define $\varphi : \mathbb{C}^* \to H$ by

$$\varphi(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

We check the four conditions of a group isomorphism:

- φ is well-defined: Suppose $a+ib = c+id$. Then $a = c$ and $b = d$ so that

$$\varphi(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \varphi(c+id).$$

- φ is injective: If $\varphi(a+ib) = \varphi(c+id)$, then

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

and so $a = c$ and $b = d$. Thus, $a+ib = c+id$, as desired.

- φ is surjective: Let $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in H$. Then $\varphi(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

- φ preserves the group operation: We have

$$\begin{aligned} \varphi((a+ib)(c+id)) = \varphi(ac-bd+i(ad+bc)) &= \begin{pmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} \\ &= \varphi(a+ib)\varphi(c+id). \end{aligned}$$

 Turn over for Exercises (2) and (3) ...

2. Let H_1 and H_2 be subgroups of G_1 and G_2, respectively. Prove that $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$.

   Solution: Let e_{G_i} denote the identity of the group G_i for $i = 1, 2$, respectively. Note that the identity of $G_1 \times G_2$ is (e_{G_1}, e_{G_2}) which is in $H_1 \times H_2$ (since $H_i \leq G_i$ for $i = 1$ and 2). Thus, $H_1 \times H_2 \neq \emptyset$. Now let (x, y) and (s, t) be two elements of $H_1 \times H_2$. Then, since H_i is a subgroup of G_i for $i = 1$ and 2, we know that $xs^{-1} \in H_1$ and $yt^{-1} \in H_2$. Hence,

   $$(x, y)(s, t)^{-1} = (x, y)(s^{-1}, t^{-1}) = (xs^{-1}, yt^{-1}) \in H_1 \times H_2.$$

   The result now follows by the One-Step Subgroup Test.

3. Complete the following definition: Let G be a group with subgroups H and K. We say that G is the *internal direct product* of H and K if

   Solution:

   - $G = HK = \{hk \mid h \in H, k \in K\}$
   - $H \cap K = \{e\}$
   - $hk = kh$ for all $h \in H$ and for all $k \in K$.