Quiz 5
Sample Solutions

Name: ____________________

Student Number: ____________________

In the space provided, please write your solutions to the following exercises. *Fully explain your reasoning.* Remember to use good notation and full sentences.

Good Luck!

1. Suppose that G is a group and let $a, b \in G$. Prove that if $|a| = m$ and $|b| = n$ with $\gcd(m, n) = 1$, then $\langle a \rangle \cap \langle b \rangle = \{e\}$.

   Solution: We have that $\langle a \rangle \cap \langle b \rangle$ is a subgroup of both $\langle a \rangle$ and $\langle b \rangle$. To see this, note that $e = a^0 = b^0 \in \langle a \rangle \cap \langle b \rangle$ and so $\langle a \rangle \cap \langle b \rangle \neq \emptyset$. In addition, if $x, y \in \langle a \rangle \cap \langle b \rangle$, then $x, y \in \langle a \rangle$ and $x, y \in \langle b \rangle$. Since $\langle a \rangle$ and $\langle b \rangle$ are groups, we know that $xy^{-1} \in \langle a \rangle$ and $xy^{-1} \in \langle b \rangle$. That is, $xy^{-1} \in \langle a \rangle \cap \langle b \rangle$. This verifies the claim that $\langle a \rangle \cap \langle b \rangle \leq \langle a \rangle$ and $\langle a \rangle \cap \langle b \rangle \leq \langle a \rangle$ (by the One-Step Subgroup Test). Thus, by the Fundamental Theorem of Cyclic Groups (from class), $|\langle a \rangle \cap \langle b \rangle|$ divides both $|\langle a \rangle| = |a| = m$ and $|\langle b \rangle| = |b| = n$. Since $\gcd(m, n) = 1$, we must have that $|\langle a \rangle \cap \langle b \rangle| = 1$ and so $\langle a \rangle \cap \langle b \rangle = \{e\}$.

 Turn over for Exercise (2)...

2. (i) Complete the following definition: Let X be a set. A permutation $\sigma \in S_X$ is a *cycle of length k* if

Solution: there exist $a_1, a_2, \ldots, a_k \in X$ such that

$$\sigma(a_1) = a_2, \sigma(a_2) = a_3, \ldots, \sigma(a_{k-1}) = a_k, \sigma(a_k) = a_1$$

and $\sigma(x) = x$ for all other elements in X.

(ii) Compute each of the following.

(iia) $(1\,3\,4\,5)(2\,3\,4)$

Solution: $(1\,3\,4\,5)(2\,3\,4) = (1\,3\,5)(2\,4)$

(iib) $[(1\,2\,3\,5)(4\,6\,7)]^{-1}$

Solution: $[(1\,2\,3\,5)(4\,6\,7)]^{-1} = (4\,6\,7)^{-1}(1\,2\,3\,5)^{-1} = (4\,7\,6)(1\,5\,3\,2)$