Quiz 4
Sample Solutions

Name: ______________________________

Student Number: ______________________________

In the space provided, please write your solutions to the following exercises. *Fully explain your reasoning.* Remember to use good notation and full sentences.

Good Luck!

1. Let G be a group and suppose that $(ab)^2 = a^2b^2$ for all a and b in G. Prove that G is an abelian group.

   Solution: For all $a, b \in G$, we have
   $$abab = (ab)^2 = a^2b^2 = aabb.$$
   Thus, by left-hand and right-hand cancellation, we have
   $$a^{-1}ababb^{-1} = a^{-1}aabbb^{-1}$$
   and so
   $$ba = ab.$$
   We conclude that G is Abelian.

 Turn over for Exercise (2)...

2. Let G be the group of 2×2 matrices with real-valued entries under addition and

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}.$$

Prove that H is a subgroup of G.

Solution: Note that H is non-empty since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in H.$$

Now let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in H$. Then $B^{-1} = \begin{pmatrix} -e & -f \\ -g & -h \end{pmatrix}$ so that

$$A + B^{-1} = \begin{pmatrix} a - e & b - f \\ c - g & d - h \end{pmatrix}.$$

Observe that, since $A, B \in H$, we have

$$(a - e) + (d - h) = (a + d) - (e + h) = 0 + 0 = 0$$

and thus $A + B^{-1} \in H$. Therefore, H is a subgroup of G.