Quiz 2
Sample Solutions

Name: ______________________________

Student Number: ______________________________

In the space provided, please write your solution to the following exercises. *Fully explain your reasoning.* Remember to use good notation and full sentences.

Good Luck!

1. Let $f : A \to B$ and $g : B \to C$ be maps.

   (i) Complete the following definition: g is *one-to-one* if

   Solution: $g(b_1) = g(b_2)$ implies that $b_1 = b_2$.

   (ii) If $g \circ f$ is onto and g is one-to-one, show that f is onto.

   Solution: Let $b \in B$. We need to show that there exists an element a in A such that $f(a) = b$. Consider $g(b) := c$. Since $g \circ f$ is onto, there exists an element a in A such that
   $$g(f(a)) = c = g(b).$$
   Since g is one-to-one, this implies that $f(a) = b$ which shows that f is onto.

2. Define a relation $\sim$ on $\mathbb{R}^2$ by stating that $(a, b) \sim (c, d)$ if and only if $a^2 + b^2 \leq c^2 + d^2$. Show that $\sim$ is transitive but not symmetric.

   Solution: We first show that $\sim$ is transitive. Suppose that $(a, b) \sim (c, d)$ and that $(c, d) \sim (e, f)$. Then
   $$a^2 + b^2 \leq c^2 + d^2 \leq e^2 + f^2.$$
   Since $a^2 + b^2 \leq e^2 + f^2$, we have that $(a, b) \sim (e, f)$ and thus $\sim$ is indeed transitive.

   To see that $\sim$ is not symmetric, consider $(1, 0)$ and $(1, 1)$ in $\mathbb{R}^2$. Since $1^2 + 0^2 \leq 1^2 + 1^2$, we have $(1, 0) \sim (1, 1)$. However, $1^2 + 1^2 \not\leq 1^2 + 0^2$ and so $(1, 1) \not\sim (1, 0)$ showing that the relation is not symmetric.

 Turn over for Exercise (3)...

3. Prove that $n! > 2^n$ for all integers $n \geq 4$.

Solution: Let $n = 4$. Then $4! = 24 > 2^4 = 16$ and so the claim is true for $n = 4$. Now assume that $k! > 2^k$ for some integer $k \geq 4$. Then

$$\begin{aligned}
(k+1)! &= (k+1)k! \\
&> (k+1)2^k \\
&\geq (4+1)2^k \\
&= 5 \cdot 2^k \\
&> 4 \cdot 2^k \\
&= 2^2 \cdot 2^k \\
&= 2^{k+2} \\
&> 2^{k+1}.
\end{aligned}$$

Therefore, by the Principle of Mathematical Induction, we conclude that $n! > 2^n$ for all integers $n \geq 4$.