MATH 2020: Algebra 1 **Dr. S. Cooper, Winter 2018**

Quiz 11
Sample Solutions

Name: ______________________________

Student Number: ______________________________

In the space provided, please write your solutions to the following exercises. *Fully explain your reasoning.* Remember to use good notation and full sentences.

Good Luck!

1. Let $\varphi : R \to S$ be a ring homomorphism. Prove that if R is a commutative ring, then $\varphi(R)$ is a commutative ring. (You may assume that $\varphi(R)$ is indeed a ring.)

   Solution: Let $x, y \in \varphi(R)$. Then there exist elements a and b in R such that $x = \varphi(a)$ and $y = \varphi(b)$. Hence,

   $$xy = \varphi(a)\varphi(b) = \varphi(ab) = \varphi(ba) = \varphi(b)\varphi(a) = yx.$$

   We conclude that $\varphi(R)$ is a commutative ring.

 Turn over for Exercise (2) ...

2. Let R be a ring.

   (i) Complete the following definition: An *ideal* I of R is

   Solution: a subring $I \subset R$ satisfying: if $a \in I$ and $r \in R$, then $ar \in I$ and $ra \in I$.

   (ii) If R is a field, show that the only two ideals of R are $\{0\}$ and R itself.

   Solution: By definition of field, R has an identity 1. As noted in class, $\{0\}$ is an ideal of R. So suppose that $I \neq \{0\}$ is an ideal of R. We have that $I \subset R$ and so it suffices to show that $R \subset I$. Let $x \in I$ be a non-zero element. Since R is a field, x must be a unit and so has a multiplicative inverse. That is, there exists a unique element $v \in R$ such that

$$xv = vx = 1.$$

   Now if $r \in R$, then

$$r = r(1) = r(xv).$$

   Since I is an ideal, $xv \in I$ and hence $r(xv) \in I$. That is, $r \in I$ and so $R \subset I$ as desired.